ISSN: 2472-3703 DOI: 10.6911/WSRJ.202406_10(6).0015

0-1 Matrices and Their Complements with Given Rank

Chao Ma[1, a]

[1]School of Science, Shanghai Maritime University, Shanghai, China

[a]machao0923@163.com

Abstract

A 0-1 matrix is an integer matrix in which each entry is either 0 or 1. The complement matrix of a 0-1 matrix A is defined by J-A, where J is the matrix in which each entry is 1. Rank of a matrix is a fundamental concept in linear algebra, which measures the maximum number of linear independent rows or columns in a matrix. We give a complete characterization of those rectangular 0-1 matrices A for which the sum of the rank of A and the rank of J-A is 1, 2 and 3. The symmetric case can also be deduced.

Keywords

0-1 matrix; Complement matrix; Rank.

1. INTRODUCTION

A 0-1 matrix is an integer matrix with each entry being 0 or 1. 0-1 matrices are closely related to combinatorial mathematics and graph theory [1-4]. They also have wide applications in probability and statistics [5-8].

Let A be an $m \times n$ 0-1 matrix. The complement matrix of A is defined and denoted by $A^c = J_{m,n} - A$, where $J_{m,n}$ is the $m \times n$ matrix with each entry being 1. Clearly the complement of a 0-1 matrix is also a 0-1 matrix. It is easy to see that A and A^c are mutually complementary, i.e., $(A^c)^c = A$.

The rank of a matrix is defined as its maximum number of linear independent rows or columns. Denote by $r(A)$ the rank of a matrix A. In [9], the authors investigated the rank relations between a 0-1 matrix and its complement. They showed that there exists an $m \times n$ 0-1 matrix A with $r(A) + r(A^c) = k$ if and only if $1 \le k \le 2\min\{m,n\}$. In this paper, we describe those $m \times n$ 0-1 matrices A satisfying $r(A) + r(A^c) = k$ for a given k. We give a complete characterization of them when $k = 1,2,3$. Then we can deduce the characterization of A in the symmetric case.

2. MAIN RESULTS

Denote by $O_{m,n}$ the $m \times n$ zero matrix. First we characterize the easy case when $r(A) + r(A^c) = 1$.

Theorem 1. Let A be an $m \times n$ 0-1 matrix. Then $r(A) + r(A^c) = 1$ if and only if $A = J_{m,n}$ or $O_{m,n}$.

Proof. If $A = J_{m,n}$ or $O_{m,n}$, then $r(A) + r(A^c) = 1$. If $r(A) + r(A^c) = 1$, then $r(A) = 1$ or 0. When $r(A) = 1$, $r(A^c) = 0$. Then $A^c = O_{m,n}$ and thus $A = J_{m,n}$. When $r(A) = 0$, $A = O_{m,n}$. This completes the proof.

ISSN: 2472-3703 DOI: 10.6911/WSRJ.202406_10(6).0015

A permutation matrix is a square matrix in which every row and every column contains a single 1 and all other entries are 0's. Two square matrices X and Y are said to be permutation equivalent if there exist permutation matrices P and Q such that $PXQ=Y$. The following result characterizes the case when $r(A)+r(A^c)=2$.

Theorem 2. Let A be an $m\times n$ 0-1 matrix. Then $r(A)+r(A^c)=2$ if and only if A is permutation equivalent to $\begin{bmatrix} J_{r,n} \\ O_{m-r,n} \end{bmatrix}$ with $0<r<m$ or $\begin{bmatrix} J_{m,s} & O_{m,n-s} \end{bmatrix}$ with $0<s<n$.

Proof. If A is permutation equivalent to $\begin{bmatrix} J_{r,n} \\ O_{m-r,n} \end{bmatrix}$ with $0<r<m$ or $\begin{bmatrix} J_{m,s} & O_{m,n-s} \end{bmatrix}$ with $0<s<n$, then $r(A)+r(A^c)=1+1=2$. If $r(A)+r(A^c)=2$, then $r(A)=r(A^c)=1$. Let α be any nonzero row of A. Since $r(A)=1$, each row β of A is either equal to α or equal to $O_{1,n}$. Note that permutation equivalent does not change the rank of a matrix. Without loss of generality, assume that the first r rows of A are equal to α, and the first s entries of α are 1's, i.e.,

$$A=\begin{bmatrix} J_{r,s} & O_{r,n-s} \\ O_{m-r,s} & O_{m-r,n-s} \end{bmatrix},$$

where $r,s>0$. We assert that $r<m,s=n$ or $r=m,s<n$. If $r=m,s=n$, then $A=J_{m,n}$ and thus $r(A)+r(A^c)=1$, which is a contraction. If $r<m,s<n$, then

$$A^c=\begin{bmatrix} O_{r,s} & J_{r,n-s} \\ J_{m-r,s} & J_{m-r,n-s} \end{bmatrix}.$$

It is easy to see that $r(A^c)=2$ and thus $r(A)+r(A^c)=3$, which is a contradiction. Therefore either A has r rows in which each entry is 1 and all other entries are 0's with $0<r<m$, or A has s columns in which each entry is 1 and all other entries are 0's with $0<s<n$. This completes the proof.

Next we characterize the case when $r(A)+r(A^c)=3$.

Theorem 3. Let A be an $m\times n$ 0-1 matrix. Then $r(A)+r(A^c)=3$ if and only if A or A^c is permutation equivalent to $\begin{bmatrix} J_{r,s} & O_{r,n-s} \\ O_{m-r,s} & O_{m-r,n-s} \end{bmatrix}$ with $0<r<m$ and $0<s<n$.

Proof. If A or A^c is permutation equivalent to $\begin{bmatrix} J_{r,s} & O_{r,n-s} \\ O_{m-r,s} & O_{m-r,n-s} \end{bmatrix}$ with $0<r<m$ and $0<s<n$, then $r(A)+r(A^c)=1+2=3$. If $r(A)+r(A^c)=3$, then $r(A)=1,r(A^c)=2$ or $r(A)=2,r(A^c)=1$.

When $r(A)=1$, let α be any nonzero row of A. Since $r(A)=1$, each row β of A is either equal to α or equal to $O_{1,n}$. Without loss of generality, assume that the first r rows of A are equal to α, and the first s entries of α are 1's, i.e.,

$$A=\begin{bmatrix} J_{r,s} & O_{r,n-s} \\ O_{m-r,s} & O_{m-r,n-s} \end{bmatrix},$$

ISSN: 2472-3703 DOI: 10.6911/WSRJ.202406_10(6).0015

where $r,s>0$. We assert that $r<m$ and $s<n$. Otherwise, $r=m$ or $s=n$ implies $r(A)+r(A^c)\le 2$, which is a contraction.

When $r(A^c)=1$, similar arguments show that A^c is permutation equivalent to $\begin{bmatrix} J_{r,s} & O_{r,n-s} \\ O_{m-r,s} & O_{m-r,n-s} \end{bmatrix}$ with $0<r<m$ and $0<s<n$. This completes the proof.

As a basic kind of matrices, symmetric matrices are very common in matrix theory, and symmetric 0-1 matrices are closely related to graphs. Finally we consider the symmetric case. By Theorem 1, we can easily deduce the case when A is a symmetric 0-1 matrix of order n satisfying $r(A)+r(A^c)=1$.

Corollary 4. Let A be a symmetric 0-1 matrix of order n. Then $r(A)+r(A^c)=1$ if and only if $A=J_{n,n}$ or $O_{n,n}$.

By Theorem 2, if a 0-1 matrix A of order n satisfying $r(A)+r(A^c)=2$, then A is permutation equivalent to $\begin{bmatrix} J_{r,n} \\ O_{m-r,n} \end{bmatrix}$ with $0<r<m$ or $\begin{bmatrix} J_{m,s} & O_{m,n-s} \end{bmatrix}$ with $0<s<n$, which implies that A is not symmetric. Thus we have the following corollary.

Corollary 5. Let A be a symmetric 0-1 matrix of order n. Then $r(A)+r(A^c)\ne 2$.

Two square matrices X and Y are said to be permutation similar if there exists a permutation matrix P such that $PXP^T=Y$. By Theorem 3, we can deduce the case when A is a symmetric 0-1 matrix of order n satisfying $r(A)+r(A^c)=3$.

Corollary 6. Let A be a symmetric 0-1 matrix of order n. Then $r(A)+r(A^c)=3$ if and only if A or A^c is permutation similar to $\begin{bmatrix} J_{r,r} & O_{r,n-r} \\ O_{n-r,r} & O_{n-r,n-r} \end{bmatrix}$ with $0<r<n$.

Proof. When A is a square 0-1 matrix of order n, by Theorem 3, there exist permutation matrices P and Q such that PAQ or PA^cQ has the form $\begin{bmatrix} J_{r,s} & O_{r,n-s} \\ O_{n-r,s} & O_{n-r,n-s} \end{bmatrix}$ with $0<r,s<n$. Since A is symmetric, we have $r=s$ and $Q=P^T$. Thus PAP^T or PA^cP^T has the form $\begin{bmatrix} J_{r,r} & O_{r,n-r} \\ O_{n-r,r} & O_{n-r,n-r} \end{bmatrix}$ with $0<r<n$. This completes the proof. □

3. CONCLUSIONS

There have been some research results on the rank relations between a 0-1 matrix and its complement matrix. And the possible values of the sum of the rank of a 0-1 matrix and the rank of its complement matrix have been given. In this paper, we further investigate the structure of those 0-1 matrices A satisfying $r(A)+r(A^c)=k$. We solve the problem in the cases when $k=1,2,3$. And we also deduce the structure of symmetric 0-1 matrices A satisfying $r(A)+r(A^c)=k$ for $k=1,3$.

ISSN: 2472-3703 DOI: 10.6911/WSRJ.202406_10(6).0015

ACKNOWLEDGEMENTS

The author would like to express his sincere thanks to referees and editor for their enthusiastic guidance and help. This paper was supported by the China Postdoctoral Science Foundation (Grant no. 2020M671086).

REFERENCES

[1] D. Petersen: Associated graphs of p-dimensional (0,1)-matrices, Ars Combin., Vol. 77 (2005), p. 143-160.

[2] A. Barvinok: On the number of matrices and a random matrix with prescribed row and column sums and 0-1 entries, Adv. Math., Vol. 224 (2010), p. 316-339.

[3] A. Kundgen, G. Leander and C. Thomassen: Switchings, extensions, and reductions in central digraphs, J. Combin. Theory Ser. A, Vol. 118 (2011), p. 2025-2034.

[4] N.J. Cavenagh: Defining sets and critical sets in (0,1)-matrices, J. Combin. Des., Vol. 21 (2013), p. 253-266.

[5] G. Louchard, H. Prodinger: Random 0-1 rectangular matrices: A probabilistic analysis, Period. Math. Hungar., Vol. 47 (2003), p. 169-193.

[6] Y. Chen, P. Diaconis, S.P. Holmes and J.S. Liu: Sequential Monte Carlo methods for statistical analysis of tables, J. Amer. Statist. Assoc., Vol. 100 (2005), p. 109-120.

[7] T. Voigt and G.M. Ziegler: Singular 0/1-matrices, and the hyperplanes spanned by random 0/1-vectors, Combin. Probab. Comput., Vol. 15 (2006), p. 463-471.

[8] P. Federbush: A mysterious cluster expansion associated to the expectation value of the permanent of 0-1 matrices, J. Stat. Phys., Vol. 167 (2017), p. 1489-1495.

[9] C. Ma and J. Zhong: Rank relations between a {0,1}-matrix and its complement, Open Math., Vol. 16 (2018), p.190-195.