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The Number Theory behind the Beauty of Apollonian Circles

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Abstract

We start from the problem of exploring the use of Descartes Formula on Curvature. In the following problems solved, we divide them into three different parts with different contexts. The bulk is to show and display our process to the audience about the law of using formula in the two-dimensional situation. We should know how we draw pictures in the area of Geometry (Curvature), what we do to compile and collect the data in the area of Statistic, and how we analyze the data in the area of Number Theory in many different ways.

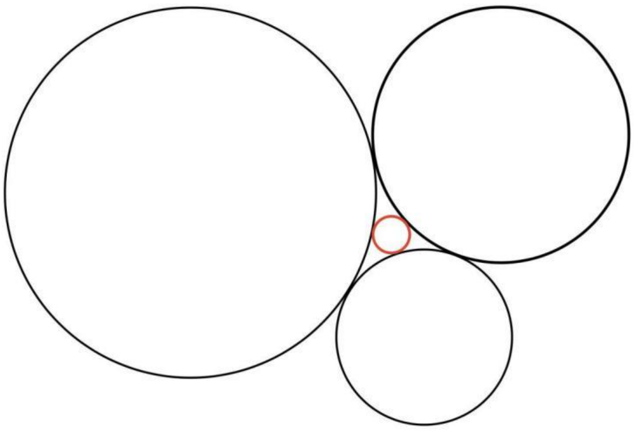
Keywords

Curvature; Number theory; Statistics method; Data analysis.

1. INTRODUCTION

Apollonian Circle Packing Problem

Drawing out the Apollonian Circles which are tangential to each other, like below.



We use the Descartes Formula on curvature to calculate different curvatures of each different circle, then we have number theory to analyze data by using graphs and tables. We already know that if k1, k2 and k3 are all integers, then we use the formula to calculate k4. If k4 is an integer too, then all the circles in this picture will have integer curvatures.

2. PREPARATION

(Part of) The Proof of Descartes Formula on Curvature

Since it is very difficult for us to prove this formula directly, we decide to use the “From special to Common” method to prove. In this situation, if there are more curvatures which are equal to 0, then it is more special. Since curvature is a really difficult part, we decide to convert the curvature.

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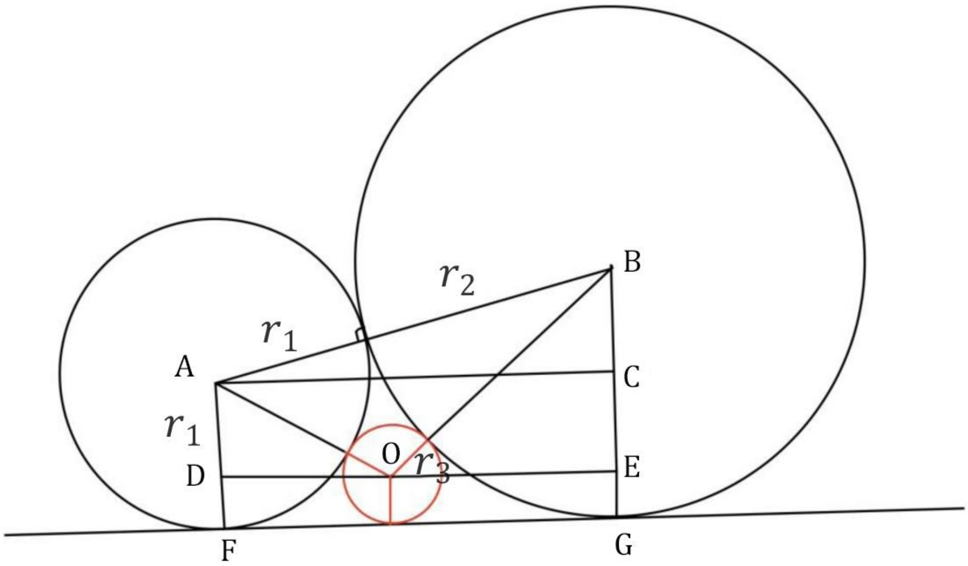
The most special situation is when three circles' curvatures are all equal to 0.

The less special situation is when two circles' curvatures are equal to 0.

Since the two situations above are too simple to prove, we will not do it here.

The below is how we prove this formula when just one curvature is equal to 0.

Let us discuss more common situations.



When $k_0 = 0, k_1 = x, k_2 = y, k_3 = z$

From the graph, we can see that $AB = r_1 + r_2, BC = r_2 - r_1$

$AD = r_1 - r_3, AO = r_1 + r_3, BE = r_2 - r_3, BO = r_2 + r_3$

In $\triangle$ ABC, $AC = \sqrt{AB^2 - AC^2} = \sqrt{4r_1r_2} = 2\sqrt{r_1r_2}$

In $\triangle$ ADO, $DO = \sqrt{AO^2 - AD^2} = \sqrt{4r_1r_3} = 2\sqrt{r_1r_3}$

In $\triangle$ BOE, $EO = \sqrt{BO^2 - BE^2} = \sqrt{4r_2r_3} = 2\sqrt{r_2r_3}$

$\because AC = DE = DO + EO$

$\therefore 2\sqrt{r_1r_2} = 2\sqrt{r_1r_3} + 2\sqrt{r_2r_3}$

$\therefore \sqrt{r_1r_2} = \sqrt{r_1r_3} + \sqrt{r_2r_3}$

$\therefore \frac{\sqrt{r_1r_2}}{\sqrt{r_1r_2r_3}} = \frac{\sqrt{r_1r_3}+\sqrt{r_2r_3}}{\sqrt{r_1r_2r_3}}$

$\therefore \frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$

So, if a is the radius of a circle

$x(t) = (a\cos t, a\sin t), 0 \le t \le 2\pi$

$x(t) = -a\sin ti + a\cos tj$

$\|x'(t)\| = \frac{ds}{dt} = a$

So that $T(t) = \frac{x\prime(t)}{\|x\prime(t)\|} = -\sin ti + \cot j$

Hence, $k = \frac{\|dT\ dt\|}{ds\ dt} = \frac{1}{a} = \|-\cos ti - sin\ tj\| = \frac{1}{a}$

Thus, we see that the curvature of a circle is always constant with value equal to the reciprocal of the radius. Therefore, the smaller the circle, the greater the curvature.

$\therefore \sqrt{k_3} = \sqrt{k_1} + \sqrt{k_2}$

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$\therefore k_3 = k_1k_2 + 2\sqrt{k_1k_2}$

Since $k_0 = 0, k_0^2 = 0$, then

$2(k_1^2 + k_2^2 + k_3^2) - (k_1 + k_2 + k_3)^2 = 0$

Change the form then we can get

$k_3^2 - 2(k_1 + k_2)k_3 + 2(k_1^2 + k_2^2) - (k_1 + k_2)^2 = 0$

We use $x = \frac{-b\pm\sqrt{4ac-b^2}}{2a}$ to calculate this quadratic function

a=1 b=-2$(k_1 + k_2)$ c=2$(k_1^2 + k_2^2) - (k_1 + k_2)^2$

By omitting the smaller root

$\therefore x = \frac{2(k_1+k_2)+\sqrt{8(k_1^2+k_2^2)-4(k_1+k_2)^2-4(k_1+k_2)^2}}{2}$

$\therefore k_3 = \frac{2(k_1+k_2)+\sqrt{8\times(2k_1+k_2)}}{2} = k_1 + k_2 + 2\sqrt{k_1k_2}$

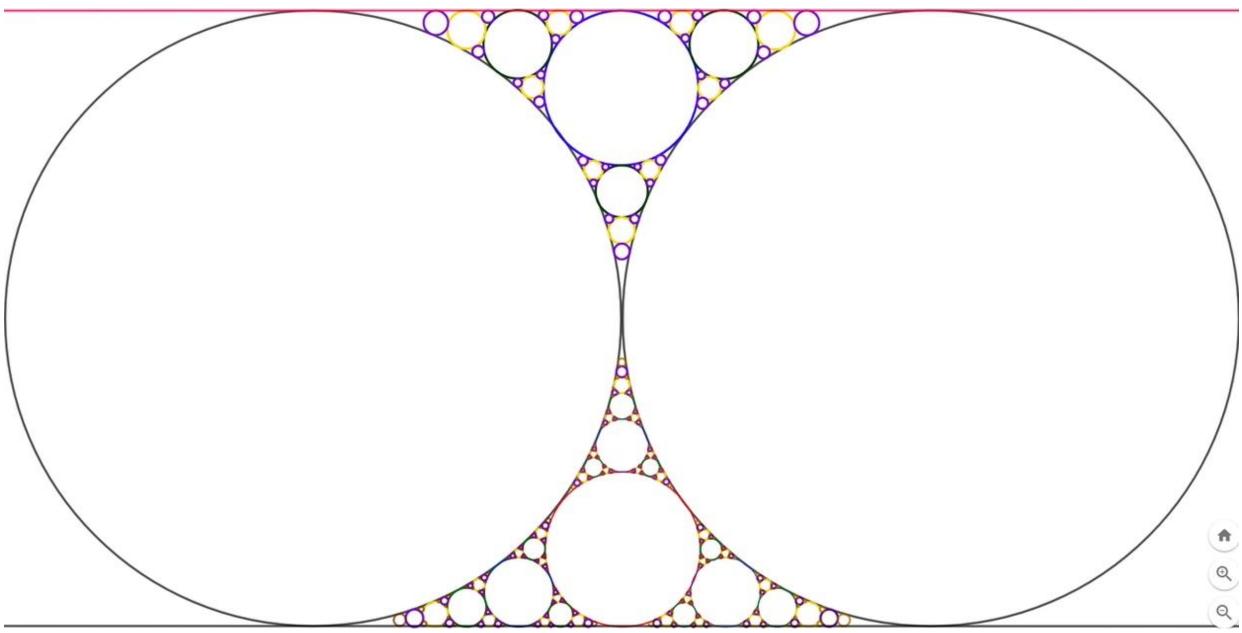
So if $k_3 = k_1 + k_2 + 2\sqrt{k_1k_2}$, then this formula is always right, and we prove that in this situation, k_3 is always equal to $k_1 + k_2 + 2\sqrt{k_1k_2}$. So the formula is right when one of the three curvatures is 0.

When it comes to the most common one: when k_1, k_2, k_3 are all not 0, you can see the proof in reference [1] at the end of this paper.

3. BULK

3.1. PART 1 the Picture

When we use the data K1=0, K2=1, K3=1, we find the result that K4 is two integers, and this satisfies the setting up of the problem. At the same time, in this situation, we will get the smallest curvature (0) which means we will find all the curvatures if we start from 0, 0 and 1. We can use the data to continue the calculation, then we get the picture below:



3.2. PART 2 How to Get the Data

From this we can see there are really a lot of different circles with different curvatures. If we want to get k4, we need to have k1, k2, and k3, then use the following formula.

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$$2(k_1{}^2+k_2{}^2+k_3{}^2+k_4{}^2)-(k_1+k_2+k_3+k_4)^2=0$$

If we want to write them out in order, we really need to use some statistics methods and define some unique and new things.

3.2.1. Firstly, x-order circle and color

To help us not omit any of the circles, we need to group the circles which are decided by when the circle appears. We first draw three original black circles, if we want to draw another circle which is tangential to three of them (represented by the red line and the red circle). We call red circles or lines the first-order circles.

Then by drawing out the first order circles, we get more chance to draw other smaller circles in the area surrounded by three big circles. We draw all these circles and we will call them second-order circles (Notation: we only do this for one time in each of the area). So and so for, we will find the third-order, forth-order, fifth-order, sixth-order circles. All these are showed in the picture. It is very easy to find out that when the order is higher, the number of circles will be larger, and also the circle will be smaller.

Color and circles:

Black: original circles (3)

Red: first-order (2)

Blue: second-order (4)

Green: third-order (12)

Orange: fourth-order (36)

Purple: fifth-order (108)

Brown: sixth-order (324)

3+2+4+12+36+108+324=489

So, there are 489 circles in the first six order in all.

3.2.2. Secondly, k4 (k1, k2, k3 (n))

In order to represent how we calculate the curvature and circle created by 3 other tangential circles, we decide to use the form above. k1, k2, k3 stand for 3 circles' curvatures, and the (n) represents how many circles have the k4 curvatures created by that particular k1, k2, k3 in that particular order (Notation: sometimes k4 will appear again and again in other order circles or with different k1, k2, k3, so that is why we emphasize two "particular").

3.2.3. Thirdly, find laws to simplify our work

Law 1

We can always get 2 root of equations since it is a quartic equation, and there are always a larger one and a smaller one. We will always just accept the larger solution which creates a smaller circle is what we need to draw and record this time.

Law 2

You may find out that the upper part of the two black circles is really similar to the lower part of the two black circles. From this we can know the n-order's Above part is exactly the same to (n-1)-order Below part.

Law 3

First order: 2 circles ($1+1=3^0+3^0$)

Second order: 4 circles ($1+3=3^0+3^1$)

Third order: 12 circles ($3+9=3^1+3^2$)

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Fourth order: 36 circles ($9+27=3^2+3^3$)

Fifth order: 108 circles ($27+81=3^3+3^4$)

Sixth order: 324 circles ($81+243=3^4+3^5$)

N-th order: $3^{n-2}+3^{n-1}$

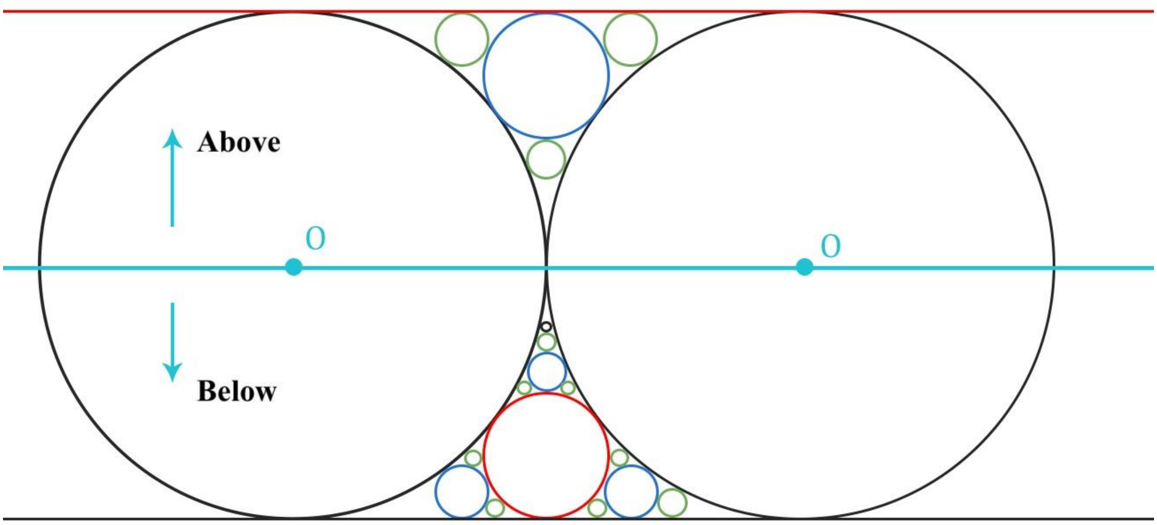
Law 4

We can know that almost every circle has its symmetrical circle. See it from the picture below. But there is one special, which is the top one of each order of Below part, and the lowest one of each order of Above part. Like k=4 (1,1,0(1)) and 24(1,1,12(1)). Then we can find the law that if k1 and k2 are both 1, then whether what k3 is, the number in the ( ) will always be 1, and all other k4 (k1,k2,k3 (n))'s n will be 2.

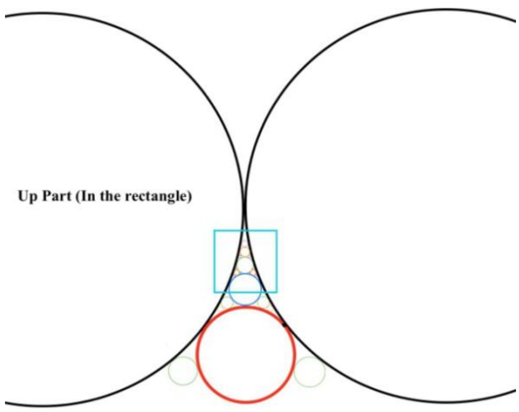
3.2.4. Fourthly, Above, Below, Up and Down

There are a lot of different circles in each order, especially when the order is very high. To make the data more in order and clearer, we decide to divide the circles from one big order into several sub-orders (Above / Below) or sub sub-orders (Up / Down).

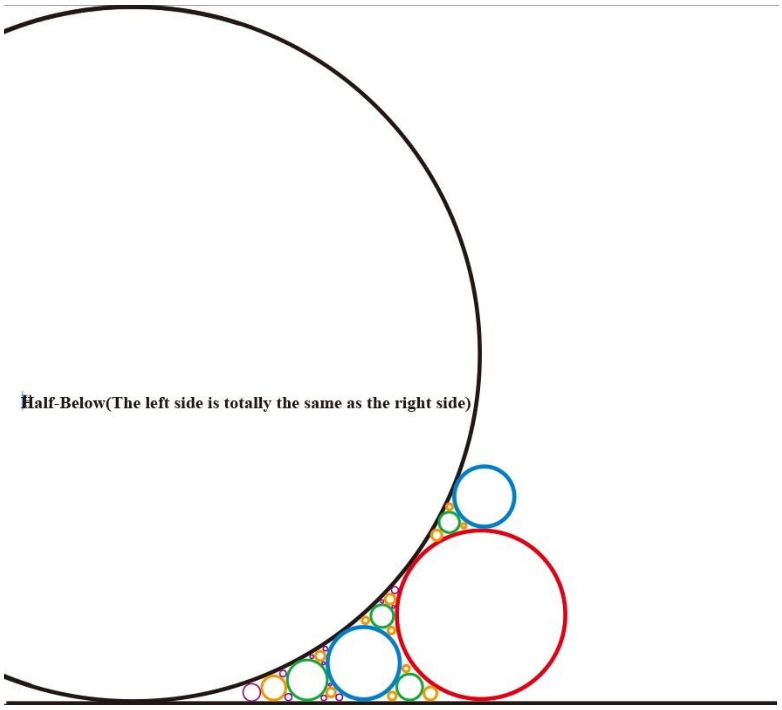
Above and Below:



Up and Down:



The Up and Down part of the Below part:



We use 4 Laws and 3 statistics methods to gather and compile all the data. Now we will just show the first four orders of the data we got.

First-order Red (2) 1+1

Above: k=0

Below: k=4

Second-order Blue (4) 1+3

Above: k=4 (1,1,0 (1))

Below: k=9 (1,4,0 (2)) 12 (1,1,4 (1))

Third-order Green (12) 3+9

Above: k=9 (1,4,0 (2)) 12 (1,1,4 (1))

Below:

Up k=24 (1,1,12 (1))

Down: k=16 (0,1,9 (2)) 28 (1,4,9 (2)) 33 (1,4,12 (2)) 25 (0,4,9 (2))

Forth-order Orange (36) 9+27

Above:

Up: k=24 (1,1,12 (1))

Down: k=16 (0,1,9 (2)) 28 (1,4,9 (2)) 33 (1,4,12 (2)) 25 (0,4,9(2))

Below:

Up: k=40 (1,1,24(1)) 73(1,12,24(2))

Down: k=25 (0,1,16(2)) 57 (1,4,28 (2)) 88 (1,12,33(2)) 52(1,9,16(2)) 97(4,12,33(2)) 64 (0,9,25(2)) 72 (1,9,28(2)) 49 (0,4,25(2)) 81 (4,9,28(2)) 49 (0,9,16(2)) 76 (4,9,25(2)) 64 (1,4,33(2))

3.3. PART 3 Analysis of the Data

We use “Number” and “Excel” to analyze the data.

How do we deal with the data?

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In Number (to analyze data).

(1)Put all the k in a sequence from small to big.

(2)Find the appearance time of each different curvature.

(3)Find the reminder when each of the curvature is divided by 3 (Modulo 3).

(4)Find the reminder when each of the curvature is divided by 4 (Modulo 4).

(5)Find all the primes of curvature k.

(6)Find the possibility that primes appear in each order (use the number of all prime curvatures of this order divided by the number of all curvatures of this order).

(7)Find the average possibility that prime numbers appear (use the number of all prime curvatures divided by the number of all curvatures).

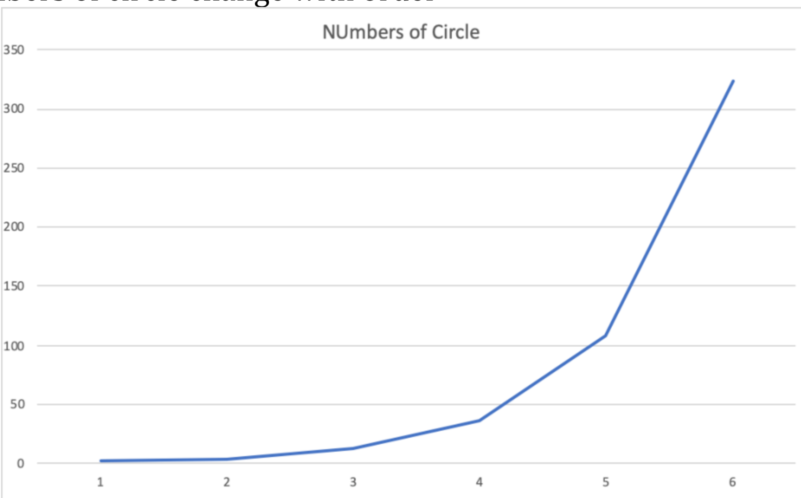
(8)Find all the square numbers of curvature k.

(9)Find prime curvatures divided by the number of all curvatures.

In Excel (to make graph).

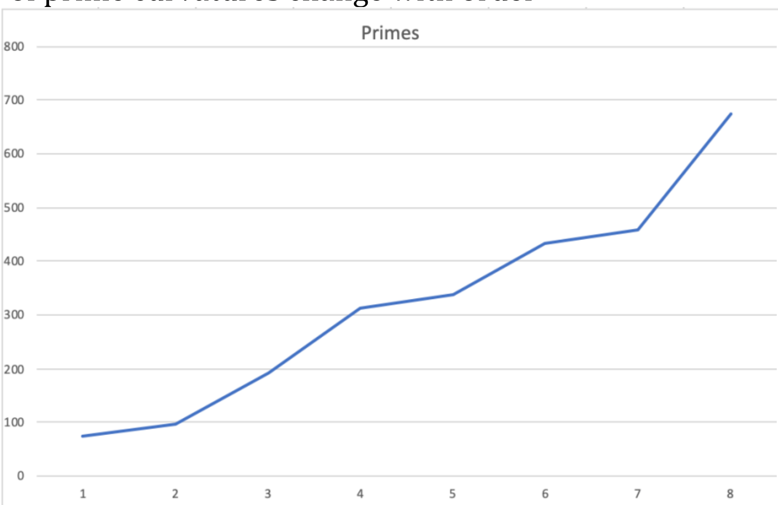
Graph 1. Numbers of circle change with order

order	NO. of Circle
1	2
2	4
3	12
4	36
5	108
6	324



Graph 2. Numbers of prime curvatures change with order

Number	Primes
1	73
2	97
3	193
4	313
5	337
6	433
7	457
8	673



Graph 3. Possibilities of appearance of prime numbers change with order + Average possibilities of appearance of prime numbers

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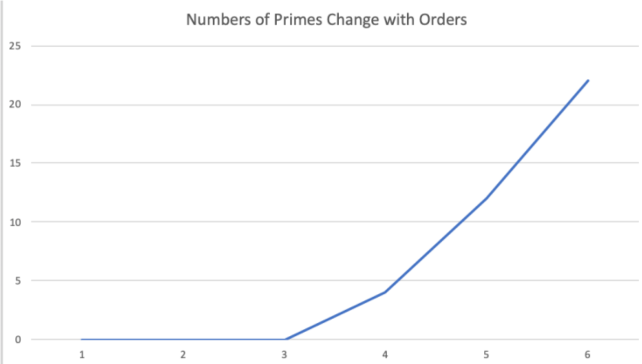
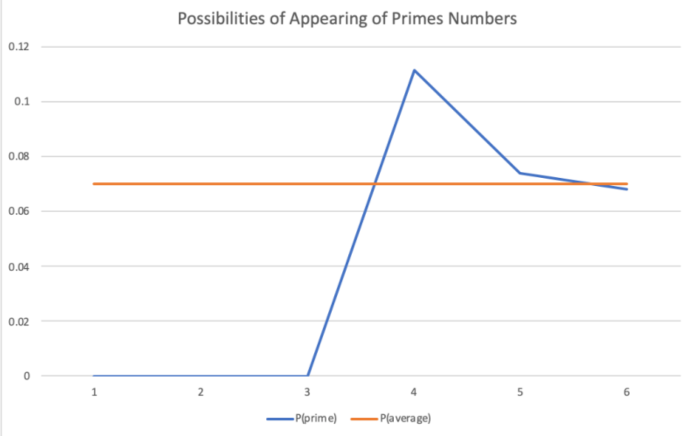
order	numbers	possibility
1	0	0
2	0	0
3	0	0
4	4	0.1111
5	12	0.0741
6	22	0.0679



P(prime)	P(average)
0	0.0699
0	0.0699
0	0.0699
0.1111	0.0699
0.0741	0.0699
0.0679	0.0699



The result we got by analyzing the data.

About Circles: the rate of change increases a lot along with the change in order (larger), and there are 109 different integer curvatures.

See Graph 1 above.

About Primes: there are 8 different prime numbers in all, and there are 34 prime curvatures.

See Graph 2 above.

When it comes to the possibilities of appearance of prime numbers, we find out that:

(1)With the increase of order, the number of prime curvatures increases a lot.

(2)The possibility of the appearance of primes first increases to the highest point when it is at the fourth-order, and then decreases continually.

(3)The first-order, second-order, third-order and sixth-order's possibilities of prime curvatures' appearance are smaller than the average P. At the fourth-order and fifth-order, the possibilities are larger than the average P.

See Graph 3 above.

About Squares: we almost find all square numbers in order 1,4,9,16 ... until 400 (20's square). We do not have enough data. We need to find more than 109 different curvatures so that we definitely find 400. You can prove it all by yourself by using "quadratic formula" of quadratic equation of the Descartes Formula.

About Modulo 3: we can find out that all the results will just 0 or 1, and there will never appear 2.

About Modulo 4: we can find out that all the results will just 0 or 1, and there will never appear 2 or 3.

4. CONCLUSION

Firstly, we try our best to first find the data in a fixed and clear way by defining different terms and also with software online to help us solve the equation.

Secondly, when we get our data, we can put them in one box, or we can decompose them into smaller boxes, or we can even further decompose them into smallest boxes. In this way and with our deep understanding of the data, we can put all the data in good and clear order without ignoring any of them.

Thirdly, we use the method as much possible as we can find to explore the sequence of K or L, while we get the idea that in which way we can find the relationship between the huge amount of data. The result is very surprising, and we also believe it can be proved in an algebraic or geometric way(as we have done before). All the things happened behind a secret of Math.

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