ISSN: 2472-3703 DOI: 10.6911/WSRJ.202010_6(10).0038

Stability of Historical Special Cases in the Three-body Problem

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Abstract

This paper introduces some special-case solutions of three-body systems and provides both analytical and simulative measures to demonstrate three-body problem. The result explores the Euler's solution, Lagrange's solution, Figure-eight solution, aiming to offer an introductory overview of the mechanisms of special circumstances and stability of three-body systems. The work picks the Lagrange's solution and Euler's solution specially by applying the simulative measures in detail. It shows that the difference between the masses has no direct correlation with the stability of the system in both Euler's and Lagrange's solution. The difference of initial positions has notable effects on the stability of the Euler's and Lagrange's solution.

Keywords

Three-body Problem, Figure-eight Solution, Lagrange's Solution, Euler's Solution, Stability.

1. INTRODUCTION

The famous three-body problem is one of the problems in the field of astrophysics that have remained mysterious and controversial for a long time. In astrophysics, and there are some general analytic solutions for centuries to solve the three-body problem since Newton proposed Newton's law of universal gravitation.

In the three-body problem, take the initial positions and velocities of three masses and solve for their subsequent motions according to Newton's laws of motion and Newton's law of universal gravitation. Compared to two-body problems, no general closed-form certain solution exists in the field of three-body problem.

To describe the three-body problems mathematically, basic Newtonian equations of motion will be applied to describe the form of directional vector $r_i = (x_i, y_i, z_i)$ indicating a 3-dimensional perspective. The three gravitationally interacting bodies are expressed in terms of m_1, m_2, m_3.

$$\ddot{r}_1 = -Gm_2 \frac{r_1 - r_2}{|r_1 - r_2|^3} - Gm_3 \frac{r_1 - r_3}{|r_1 - r_3|^3} \tag{1}$$

$$\ddot{r}_2 = -Gm_3 \frac{r_2 - r_3}{|r_2 - r_3|^3} - Gm_3 \frac{r_2 - r_1}{|r_2 - r_1|^3} \tag{2}$$

$$\ddot{r}_3 = -Gm_1 \frac{r_3 - r_1}{|r_3 - r_1|^3} - Gm_3 \frac{r_3 - r_2}{|r_3 - r_2|^3} \tag{3}$$

The total energy $\mathcal{H}$ of the three-body system is expressed in terms of momentum p.

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$$\mathcal{H} = -\frac{Gm_1m_2}{|r_1-r_2|}-\frac{Gm_2m_3}{|r_3-r_2|}-\frac{Gm_3m_1}{|r_3-r_1|}+\frac{{p_1}^2}{2m_1}+\frac{{p_2}^2}{2m_2}+\frac{{p_3}^2}{2m_3} \tag{4}$$

The work will mainly focuses on historical special-case solutions in this paper and aimsto derive the mathematical equations by importing different parameters to describe the three-body's motion in different cases and use Python to simulate the motion of different cases, varying masses and distances, in order to get a direct observation of orbits and interactions. Historical special cases of three-body problems such as Euler's solution, Lagrange's solution, "Figure-eight" solution are included in the paper.

The three-body system is highly chaotic and unpredictable. Using both analytical and simulative methods, the result still can not predict exactly movements and orbits of the three-body system. The predicted model is easily changed by its surroundings, such as interactions between the masses that are out of the original system. Many special cases of three-body problem are susceptible to small perturbations, the system can be altered permanently by unexpected forces. This paper explores whether the difference between the masses and their initial distances affect the stability of the system in Euler's solution and Lagrange's solution. After it experiences the forces out of the system, the figure-eight system will no longer remain the shape of ideal symmetrical figure-eight. The system of figure-eight system is highly unstable, so instead of using stimulative measures to calculate the life-span of the system. But there are still some interesting properties in the figure-eight system.

2. EULER'S SOLUTION

As considering the three-body problem, the paper looks at the model proposed by Euler(1765), a collinear model. Location of each mass Mi(i=1,2,3), is written as (Xi,0) ,center of mass(CM)(XG,0) Without loss of generality, assume X3<X2<X1 ,and Ri represents the position relative to CM, Ri=Xi-XG Rij=Xi-Xj and choose x=0 between M1 and M3 Therefore, R3<0 R1>0 and R1>R2>R3, define $\frac{R_{23}}{R_{12}}$=z, R13= (1+z)R12.The equation of the motion represented by the Newton equation becomes

$$\mathrm{R_1\omega^2 = \frac{GM_2}{R_{12}^2}+\frac{GM_3}{R_{13}^2}} \tag{5}$$

$$\mathrm{R_2\omega^2 = -\frac{GM_1}{R_{12}^2}+\frac{GM_3}{R_{23}^2}} \tag{6}$$

$$\mathrm{R_3\omega^2 = -\frac{GM_1}{R_{13}^2}-\frac{GM_2}{R_{23}^2}} \tag{7}$$

Of course, it gives chances to solve this system directly by using observational data about the value of masses and their distance. Observing the distance between the planets is always the easiest for human and using method of transit to calculate their masses is also familiar to researchers. However, there is one method can use to save time to collect the data from three observational distance and masses. Solve the system as shown above. In order to eliminate ω, subtract Equation (6)from Equation (5). Hence, obtain a fifth-order equation as [1]

$$(M_1+M_2)z^5+(3M_1+2M_2)z^4+(3M_1+M_2)z^3-(M_2+3M_3)z^2-(2M_2+2M_3)z-(M_2+M_3)=0 \tag{8}$$

It is obvious that z>0. According to Descartes' rule of signs—the number of positive roots either equals to that of sign changes in coefficient of a polynomial or less than it by a multiple

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of two —Equation (4) has only one positive root. With this root, z, get ω, after plugging in, which gives us the angular velocity of the system with little effort.

Interestingly, the particles move along confocal ellipses with the same eccentricity and period around the common CM. However, Euler's model has not been seen in nature because they are susceptible to small perturbations.

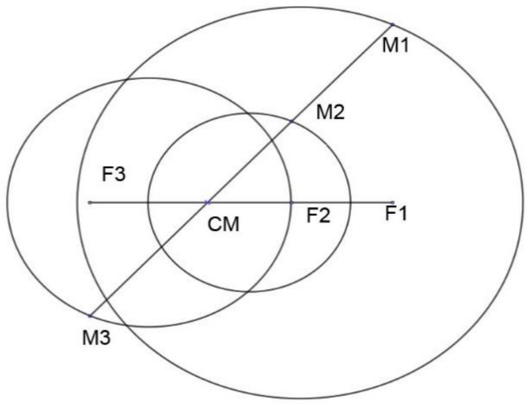

Figure 1. Euler colinear ideal situation with $m_1:m_2:m_3 = 1:2:3$

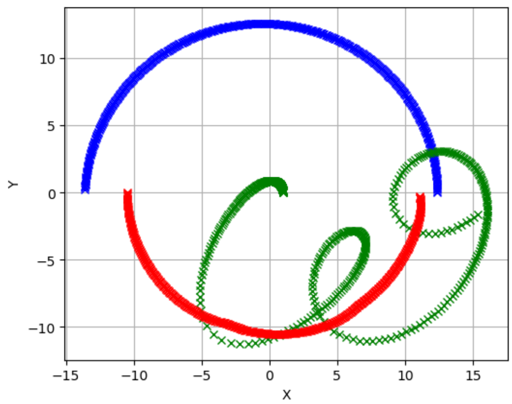

Figure 2. Simulative Euler's solution

Then using Python to check whether the difference of different masses and difference of the initial distances have a correlation with the stability of the system. Define the system is stable if they will not collide or go away from each other. The simulative measure is to use the small delta time to update their position and velocity. Nevertheless, using this "time-update" method can not calculate the life-span of a system accurately. The masses may have collided with each other, which means that their orbits are intersected once, but this moment of intersection probably is not updated by chance. In this case, using the two objects to have same coordinate in order to judge if they have collided will be inappropriate. In order to judge if they collide and go away from each other, it needs small delta time to update their position precisely. On the contrary, under the limited of simulative time, it needs large delta time to simulate the large life-span of the relatively stable system to get their exact life-span. Set up the upper simulative life-span level of 50000 seconds (approximately 13hours) to balance the accuracy and preciseness. Otherwise, simulating all situations with small delta time and no upper limit will cost tremendous time to simulate.

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First, Look at if standard deviation of the three distances between the masses affects the stability of the system. As all the masses are collinear, X_2 is defined as 1 and three masses and X_1 are derived through the random number generator in Python. Assume that all three masses revolve around the origin of the Cartesian coordinate system. As the center of mass (CM) is the origin point, then calculate for the third position, X_3. After having three initial positions and three different masses, solving three initial velocity of the masses. Define if the distance between two masses is larger than ten times R_{13}, then the masses go away from each other and the previous stable system collapse, if the distance is less than one-tenth of the R_{13}, then the masses are so closing to each other and have high probability of colliding to each other (here identify these situations as they have collided). Simulate the situations with varying masses and distance for two thousand times. The standard deviation of the distance, calculated from the initial distances of the masses, is plotted in the x-axis. The life-span(unit/seconds), based on stimulative assumption, is plotted in the y-axis as shown below.

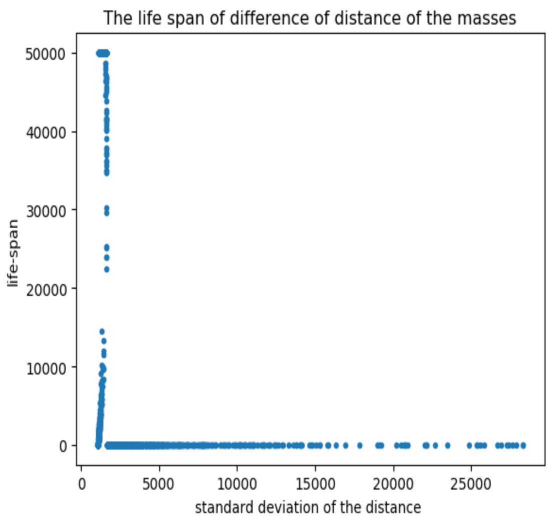

Figure 3. Influence of difference of initial distances between the masses in small scale

In Firgue 3, admit that the points in the right figure have unevenly distributed standard deviation of the distance. Control three individual distances to be generated randomly but not the standard deviation of the distance.

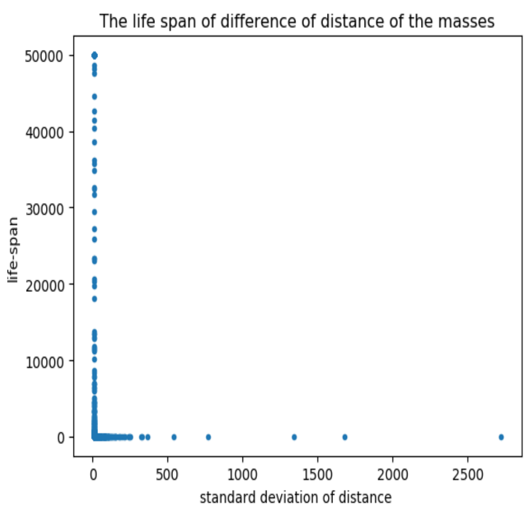

Figure 4. Influence of difference of initial distances between the masses in large scale

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Moreover, the small value of the standard deviation in the Figure 4 is caused by assumption that $X_2 = 1$. Still keeping the value of X_2 same, in the Figure3, using larger range of number generator to increase the standard deviation of the distance. According to data, the points denoted the high stability concentrate around x=0. As the difference of three distances increases, the life-span drops dramatically. The Euler's solution shows the high dependence on the difference of the initial positions of the three masses.

For the second part of simulation, fix three masses in the constant distance and vary the value of masses. Two values of masses are generated through the number generator and the third is calculated by CM. Collecting the initial position and masses, calculate their initial velocity that they revolve around the origin. Set up upper life-span level, 50000seconds, to save the time to simulate. The results are shown in Figure 5.

The simulative situations lie mainly in the upper level of life-span and near to zero life-span. The points in the middle are evenly distributed from the upper level to the zero. There's no need to enlarge the simulative upper level of life-span because conclude that there is no correlation between the standard deviation of masses in Euler's solution by visually judging the graph.

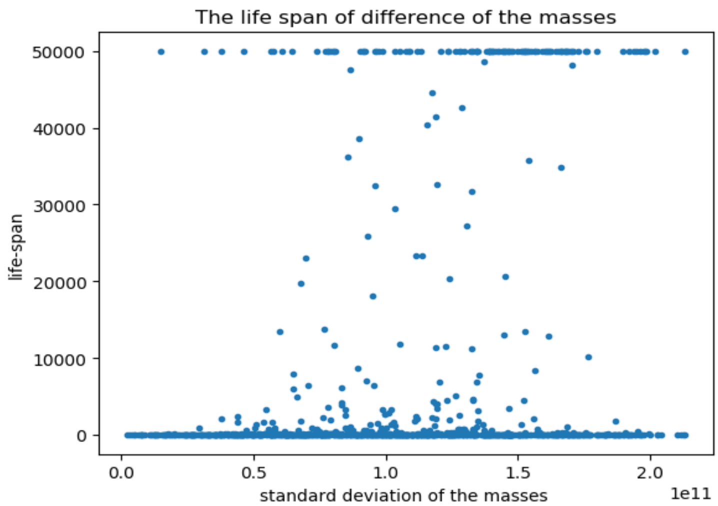

Figure 5. Life span of differenece of the masses

3. LAGRANGE'S SOLUTION

Lagrange(1772) proposed his model of three bodies by arranging three particles at the vertices of the equilateral triangle. Each particle still follows the same eccentricity and period in the shape of the eclipse with different angles. Note that the Lagrange solution is stable only if one of the three masses is much greater than the other two, also known as the restrict three-body problem. Ideally, three equal masses move in circular orbit which requires that the velocity is perpendicular to the gravitational force from CM at any time. Without loss of generality, examine one mass in particular. Its potential energy and kinetic energy are equal to

$$K(\dot{x}) = \frac{1}{2}\sum_{i=1}^{n} m_i \, |\dot{x}_i|^2 \tag{9}$$

$$U(r) = \sum_{1<i<j<n}^{n} \frac{m_i m_j}{r_{ij}} \tag{10}$$

Here import the Lagrangian, combing the equation(9) and (10) together

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$$\mathcal{L}\big(x(t),\dot{x}(t)\big) = K(\dot{x}) - U(r) \tag{11}$$

Because all gravitational forces are conservative forces which change the energy no matter which routes they take. Literally, the energy is conserved in the orbits. From the definition of Lagrangian, import the Lagrangian equation and equations about action of the path, $A(x)$.

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) = \frac{\partial\mathcal{L}}{\partial x} \tag{12}$$

$$A(x) = \int_{t_1}^{t_2} \mathcal{L}\big(x(t),\dot{x}(t)\big)\, dt \tag{13}$$

Given particular starting and ending positions, the system follows a path between the start and end points which minimizes the action of the path while keeps the energy of the system constant.

In the following discussion, in order to simplify the problem, assume that all three particles have the same mass. Each of them lies on the vertices of an equilateral triangle with the mass, m, where the sides of the triangle are equal to 2a. First, express the potential energy in Cartesian coordination

$$U(x,y) = -\frac{Gm_i m}{\sqrt{(x-a)^2+y^2}} - \frac{Gm_i m}{\sqrt{(x+a)^2+y^2}} - \frac{Gm_i m}{\sqrt{x^2+\left(y-\sqrt{3}a\right)^2}} \tag{14}$$

It makes sense that the particles are moving in the circular orbit. Therefore, put the potential energy into the form of polar coordination and parameter coordination, both of which need $\boldsymbol{\theta}$. $\rho = \frac{2\sqrt{3}}{3}a$, $(\theta,\rho) = (\omega t, \rho)$ In polar coordination, obtain

$$(\theta,\rho) = \left(\sqrt{\frac{3GM}{8a^3}}\,t, \frac{2\sqrt{3}}{3}a\right) \tag{15}$$

Plug x and y by using the parameters in trigonometric functions

$$U(\theta) = -\frac{Gm_i m}{\sqrt{\left(\rho\cos\theta - \frac{\sqrt{3}}{2}\rho\right)^2 + \left(\rho\sin\theta + \frac{\rho}{2}\right)^2}} - \frac{Gm_i m}{\sqrt{\left(\rho\cos\theta + \frac{\sqrt{3}}{2}\rho\right)^2 + \left(\rho\sin\theta + \frac{\rho}{2}\right)^2}} - \frac{Gm_i m}{\sqrt{(\rho\cos\theta)^2 + (\rho\sin\theta - \rho)^2}}$$

$$= -\frac{Gm_i m}{\rho\sqrt{\sin\theta - \sqrt{3}\cos\theta + 2}} - \frac{Gm_i m}{\rho\sqrt{\sin\theta + \sqrt{3}\cos\theta + 2}} - \frac{Gm_i m}{\rho\sqrt{2 - 2\sin\theta}} \tag{16}$$

Import the elliptical coordination to eliminate the ugly square root sign. Congregate random two masses to their center of mass(CM) in order to enable the system to have only two particles rather than three because elliptical coordination only allows the existence of two particles.[5]

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$$U(x,y) = -\frac{2Gm_i m}{\sqrt{\left(x-\frac{3}{4}\rho\right)^2+y^2}} - \frac{Gm_i m}{\sqrt{\left(x+\frac{3}{4}\rho\right)^2+y^2}} \tag{17}$$

$$\begin{cases} x = \frac{3}{4}\rho\, cos\, h\alpha\, cos\, \beta \\ y = \frac{3}{4}\rho\, sin\, h\alpha\, sin\, \beta \end{cases} \tag{18}$$

Plug x and y from equation (18) to equation (17) in elliptical coordination [2]

$$U(\alpha,\beta) = -\frac{2Gm_i m}{\frac{3}{4}\rho(cos\, h\alpha - cos\, \beta)} - \frac{Gm_i m}{\frac{3}{4}\rho(cos\, h\alpha + cos\, \beta)}$$

$$= \frac{-Gm_i m(3\, cos\, \beta + cos\, h\alpha)}{(cos\, h\alpha - cos\, \beta)(cos\, h\alpha + cos\, \beta)} \tag{19}$$

Equations (19) (16) (15) (14) have different variables. Selectively use available observational data to have enough parameters to solve the problems in the best way by combining these equations. explore the stability of Lagrange's solution by using the simulations in Python. If the three masses in the system are equal, the Lagrange's solution is stable for short period of time, but it will not last so long. Figure 6 shows the early stable stage of Lagrange's solution.

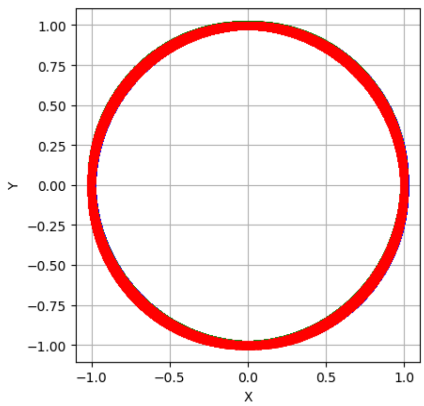

Figure 6. Early stage of Lagrange's solution

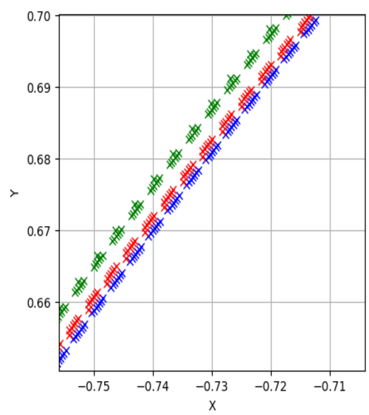

Figure 7. Detailed view of early stage

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Figure 6 is the overlooking picture of Lagrange's solution. Three orbits in different colors overlap with each other so that there is only the red color orbit in the view. After zooming in the picture, the details of their orbit. In figure 7, there is already slightly different in their orbits. Ideally, the three masses initially located at vertices of an equilateral triangle with proper direction of velocity will move in the same circular orbit. With the same radius of the orbit, their trace will be exactly overlapped in this case. However, in reality, the small difference of the radius of the circular orbit would gradually increase and finally turns out to be an unstable system. In simulations, only few situations will last for stable for years— define the stable system that the masses do not collide with each other or go away from each other.

First simulate three equal masses located at vertices of an equilateral triangle with proper direction of velocity by varying values of masses. The varying masses in the simulation are in the same order of magnitude—10^9 kg. The varying masses enable us to see the difference of the consequences of three smaller masses and three larger masses. Based on previous simulative methods, use a rough approach to judge if a system collides or goes away from each other in equilateral configuration. Define that if the distance between two masses is larger than ten times their original distance, $\mathbf{2a}$, one particle would probably go away from each other and the interactions between the masses are much smaller compared to original forces and if the distance between two masses is less than one-tenth of their original distance, it would probably collide to each other. Keeping their original distance remaining the same, simulate the situations that vary values of the masses over ten thousand times to form Figure 8 below.

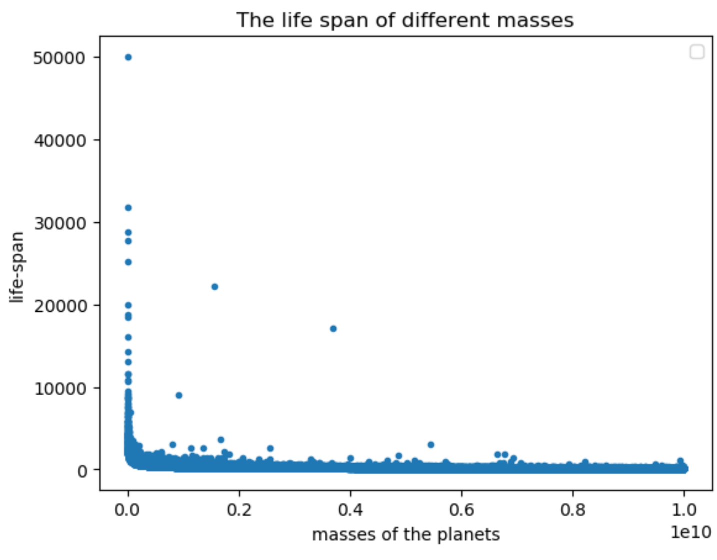

Figure 8. The life span of different masses

It clearly indicates that the systems including small masses are more stable than the systems including large masses. It makes sense that the larger masses require also larger distance to keep the system to be stable. The simulative distance, $\rho = 1$m, only is suitable for the masses in 10^9kg rather than 10^{10}kg.

The previous simulations focus on finding an optimal value of three equal masses in the system. Then vary the three masses respectively so that there is the random difference between the three masses in a system. Plot the value of standard deviation of three varying masses in the system as x value and the life-span of the system as y value in the following diagram.

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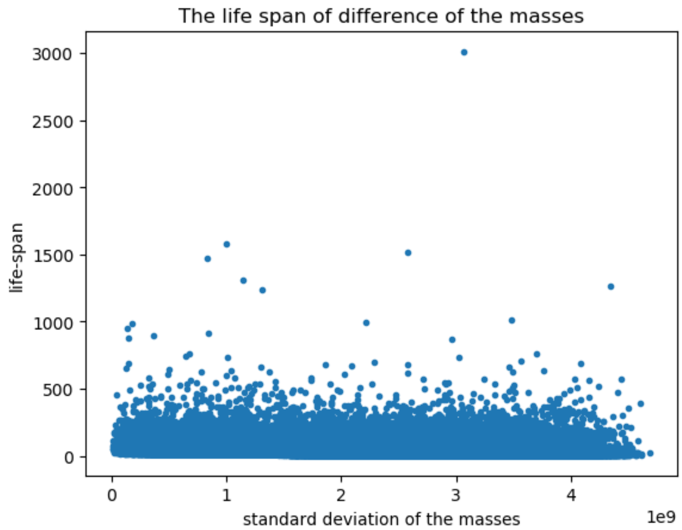

Figure 9. The life span of difference of the masses

According to the diagram above, there is no notable direct correlation with the standard deviation of the masses and the stability of the system. The points are evenly distributed for any value of standard deviation of the masses in Lagrange's solution. It should note that the simulative standard deviation of the masses is really small because all the masses are in the same order of magnitude. Lagrange(1772) also proposed the Lagrange points, which implies that the larger difference of the three masses, also known as the restrict three-body problem, can still form the stable system. Therefore, the difference of three masses in the system will not affect the stability of the system. Comparing Figure 8, it shows that three equal masses are stabler than three different masses within small standard deviation. Note that the three different masses with large standard deviation, such as the masses located in Lagrange points in the restrict three-body problem, are quite stable

Figure 10. Stimulation with small distances between the masses

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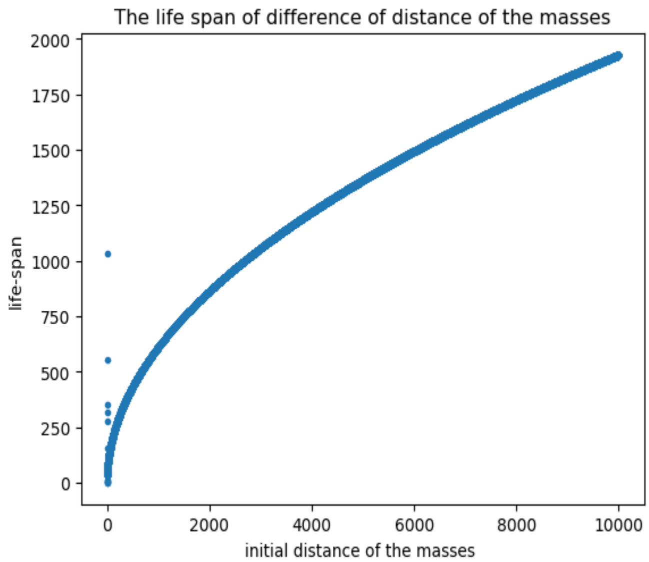

Figure 11. Stimulation with large distances between the masses

For the third part of the simulation, vary their initial distances while keeping three masses equal and constant. There is a peak around x=1, but subsequently the stability drops dramatically immediately. Where the system rises to its peak is varying from the different masses. In the simulation, pick m=2.26e9. It proves that there will always be only one optimal distance(x value of the local extreme point in the graph)for three equal masses in the system to be the stablest locally. Larger than the optimal distance, the stability of the system increases as the distances increase because their interacting gravitational forces decrease as the distance is larger. With low interacting forces, the system moves slower and neither collides soon nor goes away quickly. Results are put to use the curve_fit in Python to draw the regression analysis of Figure 11 in a mathematical expression. However, perhaps due to the effect of the peak around x=1, the result failed to draw the regression curve.

4. FIGURE-EIGHT SOLUTION

The model and stability of the figure-eight configuration were proposed by Chenciner and Montgomery [2001] [3]. The complex motion of figure-eight movements can also be broken into two simple situations which are Euler collinear situations and isosceles triangle situations as showing below

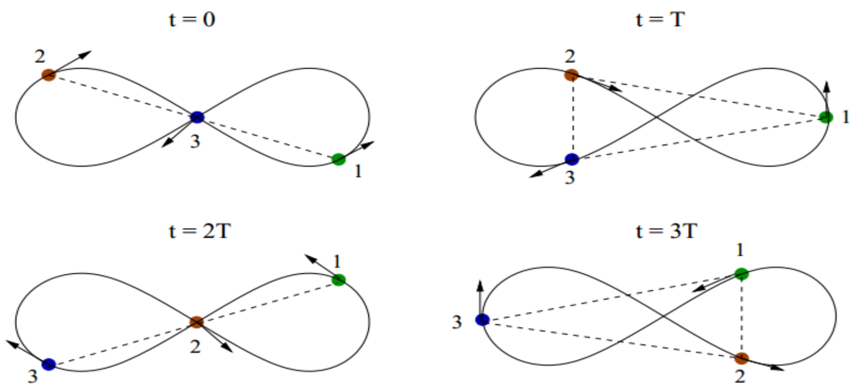

Figure 12. Figure-eight solution [4]

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In order to explore the stability of the figure-eight solution, need to consider the initial phase of three masses. [6] First of all, assume that all three particles have the same mass. If place three particles randomly on the eight-shaped orbit, the three-body system will be a chaos. Therefore, consider the system as a two-body system and put the one particle on each side to balance them. What need to do next is to place the third particles to satisfy the following equations.

The whole system can be represented as

$$x_1(t) + x_2\left(t + \frac{1}{3}T_0\right) + x_3\left(t + \frac{2}{3}T_0\right) = 0 \tag{20}$$

$$y_1(t) + y_2\left(t + \frac{1}{3}T_0\right) + y_3\left(t + \frac{2}{3}T_0\right) = 0 \tag{21}$$

These equations not only prove the great symmetry in the figure-eight movements in special cases but also show the prerequisite of the eight-figure solution.

One of the most notable features in the figure-eight solution different from the previous solution is the feature of the period. Define the eight-shaped planar position function x(0)=(0,0) for $\forall$ x ∈ ℂ, and define the intersecting point in the middle as the origin point. The whole period is defined as T, and the small interval is defined as $\boldsymbol{T_0}$ .Therefore, 12$\boldsymbol{T_0}$=T. The intervals which travel between the collinear situation and isosceles situation are equal as proved by Chenciner and Montgomery [2001] [3]. It always takes $\boldsymbol{T_0}$ to travel from the collinear situation and isosceles situation. After reaching the position of isosceles situation, three masses also take $\boldsymbol{T_0}$ to go back to the collinear situation. Assuming that all masses are equal, it gives some interesting conclusions toward the period. Their initial time can set as $\boldsymbol{0, \frac{1}{3}T_0, \frac{2}{3}T_0}$, in order to sustain their periodic motion. No matter when they travel the sum of the x-coordinate and y-coordinate is the same which equals the position of the origin point. That is

$$x(t) = \left(x_1(t), x_2\left(t + \frac{1}{3}T_0\right), x_3\left(t + \frac{2}{3}T_0\right)\right) \tag{22}$$

$$y(t) = \left(y_1(t), y_2\left(t + \frac{1}{3}T_0\right), y_3\left(t + \frac{2}{3}T_0\right)\right) \tag{23}$$

5. CONCLUSION

In summary, The results show famous historical special cases in the three-body problem. In Euler's solution, an analytical proportional number is proposed, z, to solve for ω, the angular velocity, which saves the time from solving the original three equations simultaneously by measuring three values of masses and three values of distances. Simulations of Euler's solution have proved that the difference of the masses has no direct correlation with the stability of the system and Euler's solution is stabler when the difference of initial position is small. By using parameters in polar coordination, parameter coordination, Cartesian coordination, and elliptical coordination, The results give a better understanding of the system of Lagrangian configuration. In the simulative Lagrange's solution, the standard deviation of the difference of masses also has no direct correlation with life-span of the system. For the difference of initial positions, the life-span of the system first reaches one peak of stability, followed by a straight fall to zero, and again increases gradually as the standard deviation of the initial distance increases. Moreover, the result also include the symmetrical and periodical properties in figure-eight solution.

ISSN: 2472-3703 DOI: 10.6911/WSRJ.202010_6(10).0038

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