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Crofton Type Formula for Curve Length in $\mathbb{R}^3$

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Abstract

This paper expands Crofton Theorem from $\mathbb{R}^2$ to $\mathbb{R}^3$ and discovers the relationship between the length of differentiable curves in $\mathbb{R}^3$ and the integration over the space $P(\mathbb{R}^3)$ of all planes in $\mathbb{R}^3$ through integral geometry. During the process, a curve is subdivided into segments and the integral of the number of intersection points of the segments and plane is taken in the space of all planes in $\mathbb{R}^3$. Then by Cavalieri's Principle, $I(S+\vec{r_0}) = I(S)$ (1) and $I(RS) = I(S)$ (2) are proved, where $S+\vec{r_0}$ and RS represent segment S undergoing parallel shift and rotation, respectively. Eventually, the function that includes the surface area of a sphere in $\mathbb{R}^3$ and the number of intersection points with respect to spherical coordinate is integrated, and the relation between the curve length and the integral is expressed by an integer.

Keywords

Differentiable curve; Number of intersection points; Parallel translation; Rotation; Polyline.

1. INTRODUCTION

The application of mathematics in our daily lives has been quite prevalent and common. Integral geometry studies the relationship between random variables and geometric quantities such as length and area [1]. Over a hundred years ago, Morgan Crofton came up with the Crofton Theorem, one of the fundamental results in the field of integral geometry, providing people with a formula that associates curve length and the integral of the number of intersection points of a curve and lines by an integer 2. The integral is taken over the space of all lines in $\mathbb{R}^2$.

It is known that there also exists a Crofton type formula for differentiable curves in $\mathbb{R}^3$. The goal of this research is to figure out a formula that includes the length of a differentiable curve and the integral of the number of intersection points of a segment and a plane with integral in the space of all planes in $\mathbb{R}^3$. In order to get the result, a polygonal curve is first subdivided into segments. Secondly, through Cavalieri's Principle, which explains that two solids have the same volume if two solids have the same height and the areas of their cross sections are equal at every level, it is proved that parallel translation and rotation will not change the integral of the number of intersection points of a segment and a plane with integral the space of all planes in $\mathbb{R}^3$. Then the work proves that integral with constant inside, I(kS), is equal to integral with

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constant outside, $kI(S)$, for all constant $k \in \mathbb{R}$, making it reasonable to subdivide curves into segments of any length and to integrate, respectively. Lastly, the function includes the area element on the sphere in $\mathbb{R}^3$ and the number of intersection points with respect to triple integral, each integral representing one element in spherical coordinate that is integrated. And the relation between curve length and integral is expressed by an integer. The following paragraphs demonstrate the details of how this research obtains the answers through the proving process of Crofton Theorem.

2. PROOF OF CROFTON TYPE FORMULA FOR CURVE LENGTH IN $\mathbb{R}^3$

A plane in $\mathbb{R}^3$ can be determined by a point (a,b,c) on the plane, where (a,b,c) has the shortest distance d to the origin, and a unit vector $\vec{N}$ that is orthogonal to the plane. C is a curve in $\mathbb{R}^3$, and it will be integrated over the space $P(\mathbb{R}^3)$ of all planes in $\mathbb{R}^3$. The set of all plane P having infinitely many intersection points in $P \cap C$ has measure zero [2], which means the set of points are capable of being enclosed in intervals whose total length is arbitrary small. It has a property that you can change the value of the function at points in the set without affecting the value of the integral of the function. Hence, the part of the integral over the space of all planes coming from this subset can be neglected.

2.1. Parallel Translation

As it is mentioned before, every plane in $\mathbb{R}^3$ can be described by a vector $\vec{N}$, which is perpendicular to the plane, and a scalar d, which denotes the distance from the plane to the origin. Let C be a polygonal curve in $\mathbb{R}^3$, and C is subdivided into segments. Let W_S denotes the solid in the space of all planes in $\mathbb{R}$ that consists of all planes intersecting S in exactly one point, where S represents a segment. $I(S) = \int \#(S \cap P)$ where P is a plane in $\mathbb{R}^3$ and I(S) is the integral of the number of intersection points of a segment and a plane, which is taken over the space of all planes in $\mathbb{R}^3$. Therefore, $I(S)$ is equal to the volume of W_S. Then let $S + \vec{r_0}$ denotes a parallel shift of S by a vector $\vec{r_0}$, and $I(S + \vec{r_0})$ is thus equal to the volume of $W_{S+\vec{r_0}}$. Fix a vector $\vec{N}$ in $\mathbb{R}^3$ where $\vec{N}$ is the normal vector of a particular point on the sphere S^2, and consider the subset $W_S \cap$ (all planes orthogonal to $\vec{N}$). This set is denoted by $A_{S,N}$. Similarly, $A_{S+\vec{r_0},N}$ represents the subset $W_{S+\vec{r_0}} \cap$ (all planes orthogonal to $\vec{N}$).

Cavalieri's Principle: If solids are of equal height and their corresponding cross-sections at the same level match in areas, their volumes are equal[3-5].

For solid W_S, as the planes are at the same level (height), the area of the corresponding cross-sections that are parallel to one another remain the same[3-5]. Thus, $\text{volume}(W_S) = \int \text{length}(A_{S,N})$. By the same token, $\text{volume}(W_{S+\vec{r_0}}) = \int \text{length}(A_{S+\vec{r_0},N})$. And $A_{S+\vec{r_0},N}$ is related to $A_{S,N}$ via a parallel shift by a number, which is $\frac{\vec{N}\cdot\vec{r_0}}{|\vec{N}|}$, the scalar projection of $\vec{r_0}$ to $\vec{N}$. Hence, $\text{length}(A_{S,N}) = \text{length}(A_{S+\vec{r_0},N})$. Then it is easy to conclude that $\text{volume}(W_S) = \text{volume}(W_S + \vec{r_0})$. Since $I(S) = \text{volume}(W_S)$ and $I(S + \vec{r_0}) = \text{volume}(W_{S+\vec{r_0}})$, Equation 1 is proved

2.2. Rotation

Similar to parallel translation, RS denotes a rotation of segment S with the axis of rotation l and an angle θ around axis l. As it is stated in the previous paragraph, $I(S)$ is equal to the volume of W_S and $I(RS)$ is equal to the volume of W_{RS}. Fix d in $\mathbb{R}^3$ where d is the distance from the plane to the origin, and consider the subset $W_S \cap$ (all planes orthogonal to d). This

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set is denoted by $A_{S,d}$. Similarly, $A_{RS,d}$ refers to the subset $W_{RS,d} \cap$ (all planes orthogonal to d).

For solid W_S, as the planes are at the same distance away from the origin, the corresponding cross-sections that are parallel to one another remain the same. Due to Cavalieri's Principle, $\text{volume}(W_S) = \int \text{area}(A_{S,d})$. For the same reason, $\text{volume}(W_{RS}) = \int \text{area}(A_{RS,d})$. And $A_{RS,d}$ is related to $A_{S,d}$ via a rotation around axis l by an angle θ. Hence, $\text{area}(A_{S,d}) = \text{area}(A_{RS,d})$. Then $\text{volume}(W_S) = \text{volume}(W_{RS})$ is reached. Since $I(S) = \text{volume}(W_S)$ and $I(RS) = \text{volume}(W_{RS})$, Equation 2 is proved.

2.3. Generalize to Real Numbers

Now put $I(S) = \int \#(S \cap P)$ where S represents a segment and $P \in$ (space of all planes in $\mathbb{R}^3$).

In $I(kS) = kI(S)$ (3), S stands for a segment in $\mathbb{R}^3$ and I stands for the integral of the number of intersection points of a segment and a plane. First, Equation 3 where $k \in \mathbb{Q}$ needs to be proved. Let $k = 3$, then segment S is subdivided to three equal pieces where S1=S2=S3. Since each piece is obtained by parallel shift of the other and I does not change under parallel shift according to the proof of Equation 1 in the previous paragraphs, the equation $I(S_1) = I(S_2) = I(S_3) = \frac{1}{3}I(S)$ can be gained. Moreover, since $I(S) = I(S_1) + (S_2) + I(S_3)$, it is obvious that $I(3S) = 3I(S)$. Similarly, k can be generated to any rational number $\frac{p}{q}$, such as $\frac{3}{4}$, $\frac{5}{7}$,...

In addition, rational numbers are dense in $\mathbb{R}$. To specify, if $a, b \in \mathbb{R}$ and a<b, there exists a rational number $r \in \mathbb{Q}$ such that a<r<b. Thus, $I(S)$ depends continuously on the length of S and satisfies Equation 3 for all $k \in \mathbb{Q}$. Moreover, $I(S)$ is completely defined by the x-coordinate of its right end, so $I(S)$ can be regarded as $f(x)$. Since rational numbers are dense in $\mathbb{R}$, the two continuous functions that agree on a dense set are equal, $f(kx) = kf(x)$ where k can be any real number. Hence, Equation 3 is valid when $k \in \mathbb{R}$.

2.4. Results

Any segment S in $\mathbb{R}^3$ can be changed to start at the origin and to lie in the (x,z) plane by operations of parallel shift and rotation. These operations change neither I(S) nor the length of S. Hence, in this paper's computation, it is reasonable to assume that S satisfies the segment properties. Let L be a segment in the (x,z) plane with length α and the polar coordinate $(r,0,\varphi_2)$. Segment L intersects z-axis at (0,0,z) and x-axis at $(\beta,0,0)$, where $\beta < \alpha$. r is the distance from (0,0,0) to segment L and φ_2 is the angle between x-axis and r in the (x,z) plane. r, part of the segment L, and x-axis forms a right triangle with right angle at the intersection of r and segment L and hypotenuse from (0,0,0) to $(\beta,0,0)$. According to Pythagorean Theorem, $r^2 + r^2\tan^2\varphi_2 = \beta^2$, and $\beta < \alpha$. Therefore, the following can be gained:

$$0 \le r^2 + r^2\tan^2\varphi_2 \le \alpha^2$$
$$0 \le r^2(1 + \tan^2\varphi_2) \le \alpha^2$$
$$0 \le r^2\left(\frac{\cos^2\varphi_2}{\cos^2\varphi_2} + \frac{\sin^2\varphi_2}{\cos^2\varphi_2}\right) \le \alpha^2$$
$$0 \le r^2\frac{1}{\cos^2\varphi_2} \le \alpha^2$$
$$0 \le r^2 \le \alpha^2\cos^2\varphi_2$$
$$0 \le r \le \alpha \cdot \cos\varphi_2$$

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The limit of r is from 0 to $\alpha\cos\varphi_2$. Both the limits of φ_1 and φ_2 are from 0 to π, and $r > 0$ for both $[0, \frac{\pi}{2}]$ and $[\frac{\pi}{2}, \pi]$. Thus, there are factor 2's in front of both the integral of φ_1 and φ_2. The $\sin\varphi_1$ factor is contained because it represents area element on the sphere. Also, the expression consists of number 1, which refers to the number of intersection point of segment S to all planes.

$$
\begin{aligned}
&2 \times 2\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\alpha\cos\varphi_2}\sin\varphi_1 \times 1 dr d\varphi_1 \varphi_2 \\
&= 4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\alpha\sin\varphi_1\cos\varphi_2 d\varphi_1 d\varphi_2 \\
&= 4\int_0^{\frac{\pi}{2}} - \alpha\cos\varphi_1|_0^{\frac{\pi}{2}}\cos\varphi_2 d\varphi_2 \\
&= 4\int_0^{\frac{\pi}{2}}\alpha\cos\varphi_2 d\varphi_2 \\
&= 4\alpha\sin\varphi_2|_0^{\frac{\pi}{2}} \\
&= 4\alpha
\end{aligned}
$$

Thus, $I(S) = 4\text{length}(S)$ is proved. Every differentiable curve C can be approximated by a polyline P. If the velocity vector $\vec{r}'(f) \neq 0$ at all points, then the sequence of polylines has $\text{Length}(P_n) \to \text{Length}(C)$ as n goes to infinity. From the definition of integral, $\int\int\int \#(C \cap P) = \int\int\int 4\text{length}(C)$ can be obtained. Hence, the statement of 3-dimensional Crofton Theorem in planar geometry: $\int\int\int \#(C \cap P) = \int\int\int 4\text{length}(C)$ is reached.

3. CONCLUSION

This paper generalizes the Crofton Theorem for curves in the 2-dimensional plane to the case of spatial curves, namely in the 3-dimentional plane, and discovers the relationship between the length of differentiable curves in $\mathbb{R}^3$ and the integration over the space $P(\mathbb{R}^3)$ of all planes in $\mathbb{R}^3$. The work shows that there exists a constant c such that $\int \#(C \cap P)$ equals to c times the length of curve C, and in $\mathbb{R}^3$ finds the association between the curve length and the integral by an integer 4.

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