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The Comparison of the Hofer Metric and the Continuous Metric on A Class of Poisson Manifolds

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Abstract

This paper establishes the relation between the Hofer metric and the continuous metric on the group of Hamiltonian diffeomorphisms on a class of special Poisson manifolds. With the help of the geometry structure of the Poisson manifold and the separate method of the Poisson support sets, the estimate of the Poisson displacement capacity is given, the inequality between the Hofer metric and the continuous metric is proved.

Keywords

Hofer metric, Poisson capacity, Poisson diffeomorphism.

1. INTRODUCTION

This paper is devoted to studying the relation between the Hofer metric and the continuous metric on the Poisson manifolds. Hofer metric is an important bi-invariant metric in the development of symplectic geometry. It is defined by Hofer on the Hamiltonian diffeomorphisms on the standard symplectic space [2, 3, 4, 5]. Hofer's construction is based on the minimax arguments for the action functional of the Hamiltonian systems. Later, many mathematicians are interested in starting research in this field. For example, Viterbo defined a bi-invariant metric on the Hamiltonian symplectomorphisms on the standard symplectic space by generating functions theory [9], and Polterovich, Lalonde, McDuff researched the generalization of this metric on general symplectic manifold [11, 12]. The definition of the Hofer metric is complicated and hard to compute, the non-degeneracy of this metric is not obvious. So far, there are even no results for the value of this metric on a specific Hamiltonian diffeomorphism except the identity map. Hence the extension and comparison of this metric becomes a very important topic in the studying of Hofer geometry. Banyaga defined a Hofer like metric on the group of symplectic diffeomorphisms by Hodge decomposition of symplectic isotopies on a compact symplectic manifold [7]. Han studied the extension of the Hofer metric to the identity component of symplectomorphism group, and proved that if there exists some diffeomorphism which is unbounded, then the Hofer metric does not extend to a bi-invariant metric [8]. When a symplectic map is continuously close to the identity map, does the corresponding Hamiltonian function become small enough in the continuous sense? Hofer gave an example which says that the above answer is negative [4]. The reason is the lackness of the uniformly bounded supports of the Hamiltonian diffeomorphisms. Hofer gave the following estimate of the Hofer metric and the continuous metric.

Theorem1 [4]. For every $\phi, \varphi \in D$

$$d(\phi,\varphi) \leq \text{C diameter supp}(\phi\varphi^{-1})\left|\phi - \varphi\right|_{C^0} \quad (1)$$

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Where C is a constant and less equal than 128.

Naturally, one will ask, is this result still right on the Poisson manifold? In this paper, we want to generalize this result to the Poisson manifold. Recall that Poisson manifold is a generalization of symplectic manifold. Poisson, Marsden, Weinstein are pioneers and have laid a great foundation for the development of Poisson geometry [1, 6]. There are Poisson diffeomorphism and other similar concepts in Poisson geometry. But the Poisson type invariant metric is not easy to construct because of the lackness of the variational settings. Until recently, Sun and Zhang noticed the possibility of Hofer norm on some Poisson manifold, and defined a metric needing some strong assumptions of the Hamiltonian functions on the symplectic leaves [13]. T. Rybicki found some sufficient conditions and obtained the existence of Hofer type norm [14]. After the existence of Hofer metric on the Poisson manifold, Poisson capacity was established by Sun and Liu on a class of Poisson manifold [15]. As far as we know, there are few results about the comparison on the Hofer metric and the continuous metric, we will study the relation between the two metrics on a special class of Poisson manifolds in this paper. The main theorem of the comparison results are the following:

Theorem 2. Suppose ϕ, φ are the Hamiltonian diffeomorphism on the standard Poisson space $(\mathbb{R}^n, \{\ \}_0)$, the following inequality holds:

$$d(\phi,\varphi) \le \mathrm{C}\ \ |\phi\text{-}\varphi|_{C^0} \tag{2}$$

Where C is a constant dependent on $\phi\ \varphi$.

2. PRELIMINARIES

In this section, we will give some elementary definitions in Poisson geometry, more details can be found in [1, 4, 6, 10]. First we introduce the definitions of Poisson manifold and Hofer type metric.

Definition 1 [10]. A manifold is called Poisson, if for any smooth function f, g, h on the manifold, there exists a bracket $\{\ \}$ on the functions and the following identities hold:

$$\{f, g\} = -\{g, f\}$$

$$\{f, gh\} = g\{f, h\} + h\{f, g\}$$

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0 \tag{3}$$

A diffeomorphism is called Poisson, if it keeps the Poisson bracket.

Next we recall the definition of the Hofer metric, for a compactly supported Hamiltonian function $H(t,x) \in C^\infty([0,1]\times M, \mathbb{R})$, the pseudo-norm is defined as

$$\|H_t\| = \int_0^1 osc(H_t)dt \tag{4}$$

Here the oscillation is defined as

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$$osc(H_t) = \max_x H(t,x) - \min_x H(t,x) \tag{5}$$

Definition 2[5]. Let φ be a Hamiltonian diffeomorphism, its symplectic energy is

$$E(\varphi) = \inf\{\|H_t\| \,|\, \phi_H^1 = \varphi\} \tag{6}$$

For $f \in C^\infty(M)$, we define

$$\|f\| = \inf\{\|f_1\| \,|\, f = f_1 + f_2, f_2 \in Cas(M)\} \tag{7}$$

Here $f_2 \in C^\infty(M)$ is a Casimir function. Replaced the norms in (6) by (7), if we assume that the restrictions of the Hamiltonian functions on the symplectic leaves are well defined, we can get the Hofer type norm on the Poisson manifold. In the following we will make this assumptions for the manifold.

Definition 3 [15]. The standard Poisson space $(\mathbb{R}^n, \{\ \}_0)$ is defined as following:

(1) If $n = 2m$, the standard Poisson manifold is just the standard symplectic space $(\mathbb{R}^{2m}, \omega_0)$.

(2) If $n = 2m+1$, for every point in the the 1dimensional space $\mathbb{R}$, the symplectic leaf is $2m$ dimensional subspace with the standard symplectic form ω_0, the restriction of the Poisson bracket on the left space $\mathbb{R}$ is 0.

3. PROOF OF MAIN RESULTS

In this part we can give the proof of the comparison results on a class of Poisson manifold. We first recall the following definition which is similar with the symplectic case.

Definition 4. We call sets $S_1, \cdots, S_k$ on the standard Poisson space properly separated, if for any $A_j \subset S_j$ of bounded subsets, there exists a Poisson diffeomorphism τ and numbers $a_1 \prec a_2 \cdots \prec a_{k-1}$ such that $\tau(A_j)$ are separated by the hyperplanes $\{x_1 = a_j\}$.

We now have the following lemma:

Lemma 5. Assume that $S_1, \cdots, S_k$ on the standard Poisson space are properly separated, then given $R \geq 0$, there exists a Poisson diffeomorphism τ such that $\mathrm{dist}(\tau(S_i), \tau(S_j)) \geq R$.

Proof: If $n = 2m$, this is the symplectic space, by the results of Lemma 8 in P175 of [4], we have the results. If $n = 2m+1$, we assume that the hyperplanes are parallel to the last axis, since the last axis is transversal with each symplectic leaf. Following the procedure of Lemma 8 we finish the proof.

Lemma 6. Let $0 = a_1 \prec a_2 \cdots \prec a_k, v_j = a_j e_1 \in \mathbb{R}^{2n+1}$ $e_1 =$ (1, 0, ⋯, 0). Let

$$S_j^t = S_j + t v_j$$

$$H(t, x) = a_j y_1 \tag{8}$$

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If $(x, y) \in S_j^t$, using the Lemma 8 in P175 of [4] again, we know that on the leaf level, the restriction of maps φ_H^t acts on each sets by the symplectic way, and the distances between the sets S_j^t are larger than tR. This finishes the proof.

Lemma 7. Assume $\psi_1, \psi_2, \cdots, \psi_k$ are Hamiltonian diffeomorphisms and have properly separated supports. Then we have

$$E(\psi_1\psi_2\cdots\psi_k) \leq c\max_j E(\psi_j) \tag{9}$$

Here c is a large positive constant.

Proof: According to the proposition in [13], the Poisson energy is invariant under conjugation, we now use a trick as that in Lemma 9 of P176 in [4], due to the Poisson structure of the standard Poisson space, let $\psi_j = \varphi_{H_j}$ and S_j be the support. By lemma 8, define $\tilde{\psi}_j = \tau\psi_j\tau^{-1}$, the support of $\tilde{\psi}_j$ are disjoint and we have the following identity

$$E(\psi_1\psi_2\cdots\psi_k) = E(\tilde{\psi}_1\tilde{\psi}_2\cdots\tilde{\psi}_k) \tag{10}$$

So the Hamiltonian function of $\tilde{\psi}_1\tilde{\psi}_2\cdots\tilde{\psi}_k$ satisfies

$$\tilde{H} = \tilde{H}_1 + \tilde{H}_2 + \cdots + \tilde{H}_k \tag{11}$$

By the definition of the energy, we get the result.

Recall the Poisson capacity is defined as following

$$c(M) = \sup_{l_\alpha} c_s(l_\alpha) \tag{12}$$

Here $l_\alpha \in M$ are the symplectic leaves, c_s is the symplectic capacity. We know that this is a Poisson capacity on standard Poisson space.

Lemma 8. If the Hamiltonian function satisfies supp $(H) \subset [0,1] \times U$, then the energy satisfies:

$$E(\varphi_H) \leq C e_p(U) \tag{13}$$

Here $e_p(U)$ is the proper displacement energy, C is a large positive constant.

Having this lemma we now need the following estimate.

Lemma 9. Let $\psi \neq \mathrm{id}$ be a Hamiltonian diffeomorphism, and $|\psi\text{-id}|_{C^0} \leq \delta$. For every open domain in the standard Poisson space, if supp $\psi \bigcap Q \neq \varnothing$, there exists a Hamiltonian diffeomorphism φ and positive constant satisfying

$$\varphi|_Q = \psi|_Q$$

$$\mathrm{supp}\,\varphi \subset U$$

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$$E(\varphi) \leq Ce_p(U) \tag{14}$$

Lemma 10. Let $\psi \neq \mathrm{id}$ be a Hamiltonian diffeomorphism, supp $\psi \subset U$. If $U = (a_1, a_2) \times (b_1, b_2) \oplus R^{2n-1}$, then

$$E(\varphi_H) \leq C(a_1, a_2) \times (b_1, b_2) \tag{15}$$

This lemma is a consequence of the above lemma.

Proof of the main Theorem: Now we give the outline of the proof. We just consider the case for $n = 2m+1$, the Poisson coordinates be $(x_1, x_2, \cdots x_m, y_1, y_2 \cdots y_m, z_{2m+1})$, by the geometry structure of the standard Poisson space, we know that the leaves are parallel $2m$ dimensional standard symplectic space. For a Hamiltonian diffeomorphism $\psi \neq \mathrm{id}$, let a number δ satisfy $|\psi\text{-id}|_{C^0} \leq \delta$ then set

$$Q_k = (a_k - \varepsilon, a_k + \varepsilon) \times R \oplus R^{2n-1} \tag{16}$$

Then we use above lemmas and the estimate for the separated support methods for Q_k and $E(\varphi_k)$, we can get the result.

Remark 1. The key point in our proof is the structure of the standard Poisson space. We reduce the Poisson question to the standard symplectic leaf. For other Poisson bracket, it is complicated to compute the estimation for the displacement energy.

Remark 2. From the construction we get that, when a Poisson map is continuously close to the identity map, the corresponding Hamiltonian function not necessarily becomes small enough in the continuous sense. The example can be defined by

$$\tilde{H}_n(x, y, z) = \frac{1}{n} f_n(x) \chi_n(x, y) \tag{17}$$

On the 3 dimensional standard symplectic space, $f_n(x)\chi_n(x, y)$ are defined similarly with Hofer in symplectic 2 space.

4. CONCLUSION

In this paper we construct an inequality between the Hofer type metric and the continuous metric on the standard Poisson space. Using the properly separated method for the Hamiltonian diffeomorphisms and the standard symplectic leaf structure, we give some estimation for the compound of the Hamiltonian diffeomorphisms and establish the relation between the energy and the displacement energy on the standard Poisson space. This inequality answers the question of extension from symplectic to Poisson for the comparison result of the Hofer metric.

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