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The Application of Finite Element Method in 2D MT Forward Modeling Calculation

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Abstract

Finite element method is a common method in MT forward modeling. This paper mainly studies the application of triangulation finite element method in forward modeling; The steps of forward modeling with finite element method are as follows: (1) Constructing the expression of variational formulation from the electromagnetic field differential equation; (2) Divide the solution area into several small triangle elements; (3) Using shape function to write out the matrix of functional expression corresponding to each unit; (4) Synthesize the population matrix and solve it to get the field value.

Keywords

MT forward; Finite element method; Triangulation; Variational formulation; Form function.

1. INTRODUCTION

There are three forward modeling methods for magnetotelluric sounding(MT): analytical method, model experiment method and numerical method. The numerical method is commonly used in practical work, and the finite element method is commonly used in numerical method. The finite element method was first developed in the field of elasticity in the 1950s. It can be well applied to the field of complex shape. Its efficiency is high. In recent years, with the improvement of computer technology, its application is more and more extensive. This paper mainly discusses the application of finite element method in 2D MT forward calculation.

2. BASIC THEORY

2.1. MT Theory

The theoretical basis of magnetotelluric sounding is Maxwell's equations, which correspond to harmonic field:

$$\begin{cases} \nabla \times E = i\mu\omega H \\ \nabla \times H = \sigma E \\ \nabla \cdot E = 0 \\ \nabla \cdot H = 0 \end{cases} \tag{1}$$

Take the curl on both sides of the first and second expressions of formula (1):

$$\begin{cases} \nabla \times \nabla \times E = i\mu\omega(\nabla \times H) \\ \nabla \times \nabla \times H = \sigma(\nabla \times E) \end{cases} \tag{2}$$

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The vector wave equation of electromagnetic field can be obtained by combining the vector analysis formula (2) and the first and second expressions of (1):

$$\begin{cases} \nabla^2 E - k^2 E = 0 \\ \nabla^2 H - k^2 H = 0 \end{cases} \tag{3}$$

In expression (3), k is complex wave number, $k = \sqrt{-i\omega\mu\sigma - \omega^2 \mu\varepsilon} \approx \sqrt{-i\omega\mu\sigma}$.

2.2. Two Dimensional Forward Model

When the resistivity of underground medium changes in two directions, take the strike direction of geological structure as z-axis, Y-axis vertical upward, X-axis vertical Z-axis and in a plane, as shown in figure 1.

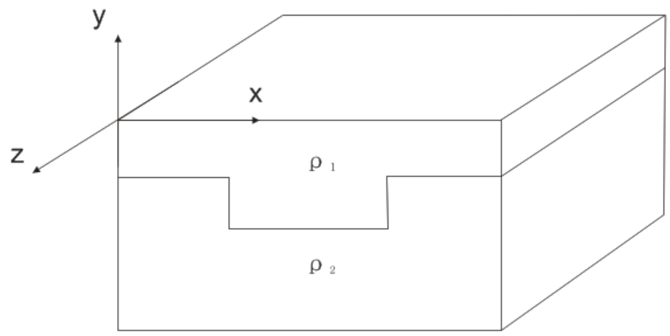

Figure 1. Two dimensional forward model

According to the Maxwell equations, the electromagnetic field equation with an angular frequency of ω is:

$$\nabla \times E = i\omega\mu H \tag{4}$$

$$\nabla \times H = (\sigma - i\omega\mu)E \tag{5}$$

The differential equations of TE polarization mode and TM polarization mode can be obtained by expanding (4) and (5).

TE mode:

$$\begin{cases} \dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y} = (\sigma - i\omega\varepsilon)E_z \\ \dfrac{\partial E_z}{\partial y} = i\omega\mu H_x \\ -\dfrac{\partial E_Z}{\partial x} = i\omega\mu H_Y \end{cases} \tag{6}$$

TM mode:

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$$\begin{cases} \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega\varepsilon H_z \\ \frac{\partial H_Z}{\partial y} = (\sigma - i\omega\varepsilon)E_X \\ -\frac{\partial H_Z}{\partial x} = (\sigma - i\omega\varepsilon)E_y \end{cases} \tag{7}$$

From the second and third expressions in (34) and (35), we can solve HX, hy, ex and ey, and substitute them into the first one respectively, and we can get the following partial differential equations:

$$\begin{cases} \frac{\partial}{\partial x}(\frac{1}{i\omega\mu}\frac{\partial E_Z}{\partial x}) + \frac{\partial}{\partial y}(\frac{1}{i\omega\mu}\frac{\partial E_Z}{\partial y}) + (\sigma - i\omega\varepsilon)E_Z = 0 \\ \frac{\partial}{\partial x}(\frac{1}{\sigma - i\omega\varepsilon}\frac{\partial H_Z}{\partial x}) + \frac{\partial}{\partial y}(\frac{1}{\sigma - i\omega\varepsilon}\frac{\partial H_Z}{\partial y}) + i\omega\mu H_Z = 0 \end{cases} \tag{8}$$

(8) Can be expressed as follows:

$$\nabla \cdot (\tau \nabla u) + \lambda u = 0 \tag{9}$$

TE mode:

$$u = E_Z, \quad \tau = \frac{1}{i\omega\mu}, \quad \lambda = \sigma - i\omega\varepsilon \tag{10}$$

TM mode:

$$u = H_Z, \quad \tau = \frac{1}{\sigma - i\omega\varepsilon}, \quad \lambda = i\omega\mu \tag{11}$$

2.3. Boundary Conditions

After deduction, the boundary conditions of magnetotelluric sounding method are as follows:

$$\begin{cases} \nabla \cdot (\tau \nabla u) + \lambda u = 0 & \in \Omega \\ u = 1 & \in AB \\ \frac{\partial u}{\partial n} = 0 & \in AD, BC \\ \frac{\partial u}{\partial n} + ku = 0 & \in CD \\ u_1 = u_2 & \in \Gamma_1 \\ \tau_1 \frac{\partial u_1}{\partial n} = \tau_2 \frac{\partial u_2}{\partial n} & \in \Gamma_1 \end{cases} \tag{12}$$

2.4. Variational Formulation

We assume that there is only one inhomogeneous body in the region, and construct the functional:

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$$
\begin{aligned}
I(u) &= \int_{\Omega}[\frac{1}{2}\tau(\nabla u)^2 - \frac{1}{2}\lambda u^2] \\
&= \int_{\Omega}[\frac{1}{2}\tau_1(\nabla u_1)^2 - \frac{1}{2}\lambda_1 {u_1}^2] + \int_{\Omega}[\frac{1}{2}\tau_2(\nabla u_2)^2 - \frac{1}{2}\lambda_2 {u_2}^2]
\end{aligned} \tag{13}
$$

The variation expression of (13) is as follows:

$$
\begin{aligned}
\delta I(u) &= \int_{\Omega 1}\tau_1\nabla u_1\cdot\nabla\delta u_1 d\Omega - \int_{\Omega 1}\lambda_1 u_1\delta u_1 d\Omega + \int_{\Omega 2}\tau_2\nabla u_2\cdot\nabla\delta u_2 d\Omega - \int_{\Omega 2}\lambda_2 u_2\delta u_2 d\Omega \\
&= \int_{\Omega 1}\nabla\cdot(\tau_1\nabla u_1\delta u_1)d\Omega - \int_{\Omega 1}[\nabla\cdot(\tau_1\nabla u_1) + \lambda u_1]\delta u_1 d\Omega \\
&+ \int_{\Omega 2}\nabla\cdot(\tau_2\nabla u_2\delta u_2)d\Omega - \int_{\Omega 2}[\nabla\cdot(\tau_2\nabla u_2) + \lambda u_2]\delta u_2 d\Omega
\end{aligned} \tag{14}
$$

Combined with the boundary condition (12), the variational formulation corresponding to the boundary value problem can be obtained as follows:

$$
\begin{cases}
F(u) = \int_{\Omega}[\frac{1}{2}\tau(\nabla u)^2 - \frac{1}{2}\lambda u^2]d\Omega + \int_{CD}\frac{1}{2}\tau k u^2 d\Gamma \\
\delta F(u) = 0 \\
u\,|_{AB} = 1
\end{cases} \tag{15}
$$

2.5. Finite Element Method For Solving Variational Formulation

The triangulation diagram of finite element method is as figure 2:

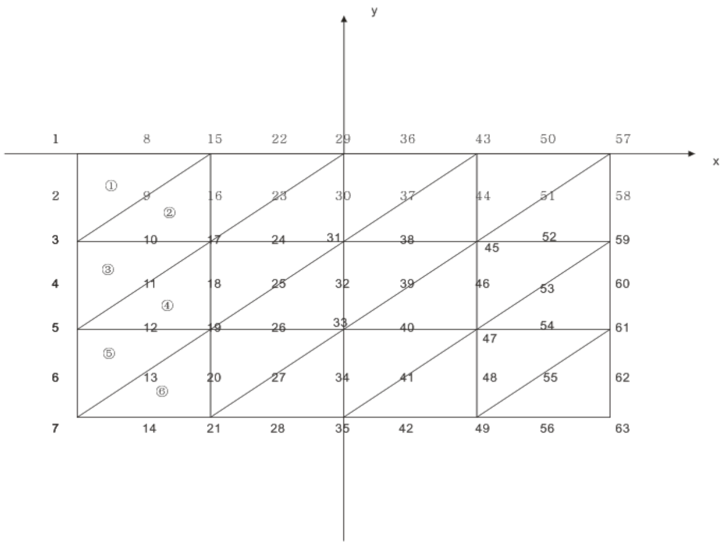

Figure 2. Triangulation diagram

As shown in the figure, the underground half space is divided into rectangles, and then each rectangle is divided into two triangles. For a single triangle, three vertices and the midpoint of each edge are selected as nodes. The node numbers are arranged from the top to the bottom as

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shown in the figure, and continue to be arranged from the top of the second column after arriving at the bottom node. The cell numbers are also arranged in this way (for simplicity, only the nodes are listed First column).

Analyze the integral expression in each cell, and solve the first integral of the first formula of formula (50) on the cell:

$$\int_{\Omega}\frac{1}{2}\tau(\nabla u)^2 d\Omega = \frac{1}{2}u_e^T(k_{ij})u_e = \frac{1}{2}u_e^T K_e u_e \tag{16}$$

The integral of triangle element is as follows:

$$\iint_{\Delta}(\nabla u)^2 = \iint_{\Delta}[(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2]dxdy = u^T ku \tag{17}$$

Where Kij represents the elements of row I and column J in the element matrix, and the elements of the element matrix are:

$k_{11} = \tau(3a_1^2 + 3b_1^2)/(12\Delta), \quad k_{21} = \tau(-a_1a_2 - b_1b_2)/(12\Delta),$

$k_{22} = \tau(3a_2^2 + 3b_2^2)/(12\Delta)$

$k_{31} = \tau(-a_1a_3 - b_1b_3)/(12\Delta), \quad k_{32} = \tau(-a_2a_3 - b_2b_3)/(12\Delta),$

$k_{33} = \tau(3a_3^2 + 3b_3^2)/(12\Delta)$

$k_{41} = 0, \quad k_{42} = \tau(4a_2a_3 + 4b_2b_3)/(12\Delta),$

$k_{43} = \tau(4a_2a_3 + 4b_2b_3)/(12\Delta)$

$k_{44} = \tau(8(a_2^2 - a_1a_3) + 8(b_2^2 - b_1b_2))/(12\Delta),$

$k_{51} = \tau(4a_1a_3 + 4b_1b_3)/(12\Delta)$

$k_{52} = 0, \quad k_{53} = \tau(4a_1a_3 + 4b_1b_3)/(12\Delta),$

$k_{54} = \tau(8a_1a_2 + 8b_1b_2)/(12\Delta)$

$k_{55} = \tau(8(a_3^2 - a_1a_2) + 8(b_3^2 - b_1b_2))/(12\Delta),$

$k_{61} = \tau(4a_1a_2 + 4b_1b_2)/(12\Delta)$

$k_{62} = \tau(4a_1a_2 + 4b_1b_2)/(12\Delta), \quad k_{63} = 0,$

$k_{64} = \tau(8a_1a_3 + 8b_1b_3)/(12\Delta)$

$k_{65} = \tau(8a_2a_3 + 8b_2b_3)/(12\Delta), \quad k_{66} = \tau(8(a_1^2 - a_2a_3) + 8(b_1^2 - b_2b_3))/(12\Delta)$

After calculating the matrix, multiply τ to represent formula (51).

The second integral is:

$$\int_{e}\frac{1}{2}\lambda u^2 d\Omega = \frac{1}{2}u_e^T K_{2e} u_e \tag{18}$$

The element matrix corresponding to (18) is (because of its symmetry, only the lower part is listed):

$$k_{2e} = \frac{\lambda\Delta}{180}\begin{bmatrix} 6 & & & & & \\ -1 & 6 & & & & \\ -1 & -1 & 6 & & & \\ -4 & 0 & 0 & 32 & & \\ 0 & -4 & 0 & 16 & 32 & \\ 0 & 0 & 4 & 16 & 16 & 32 \end{bmatrix} \tag{19}$$

When the program traverses to the bottom unit, we should add the third term of integration:

$$\int_{CD} \frac{1}{2}\tau k u^2 d\Gamma = \frac{1}{2} u_e^T k_{3e} u_e \tag{20}$$

The corresponding element matrix is (the node number falling on the CD side is 2-4-3):

$$k_{3e} = \frac{\tau k l}{30}\begin{bmatrix} 0 & & & & & \\ 0 & 4 & & & & \\ 0 & 2 & 16 & & & \\ 0 & -1 & 2 & 4 & & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

Select TE or TM polarization modes, and calculate the corresponding τ and λ, then traverse all the divided units, add up the unit matrix of each unit, and store it in the total matrix which include all the unit matrices according to the number of each node, finally we can get the total matrix (total coefficient matrix). Substituting boundary conditions into coefficient matrix and solving the system of equations, we can get the field value U. if it is TE mode, U represents the electric field E. If it is TM mode, U represents the magnetic field H. Finally, the apparent resistivity and phase can be calculated by U

The algorithm flow chart is as follows:

3. CALCULATION RESULTS

The object of this program is G-type model of 2-layer geoelectric section, which is calculated by TE model. The frequency is 10Hz, the thickness of the first layer is h1 = 1000m, the resistivity of the first layer is ρ1 = 100 Ω·m, and the second layer is ρ2= 1000 Ω·M. In this paper, We use triangulation, biquadratic interpolation. The number of intervals in X direction and Y direction are both 24, and the width and height of each interval are both 200m. So the total number of elements is 1152, and the total number of nodes is 2401. The field value E can be obtained by inputting each data into the program, and then the apparent resistivity value and impedance phase value of the earth surface can be calculated, as shown in figure 4:

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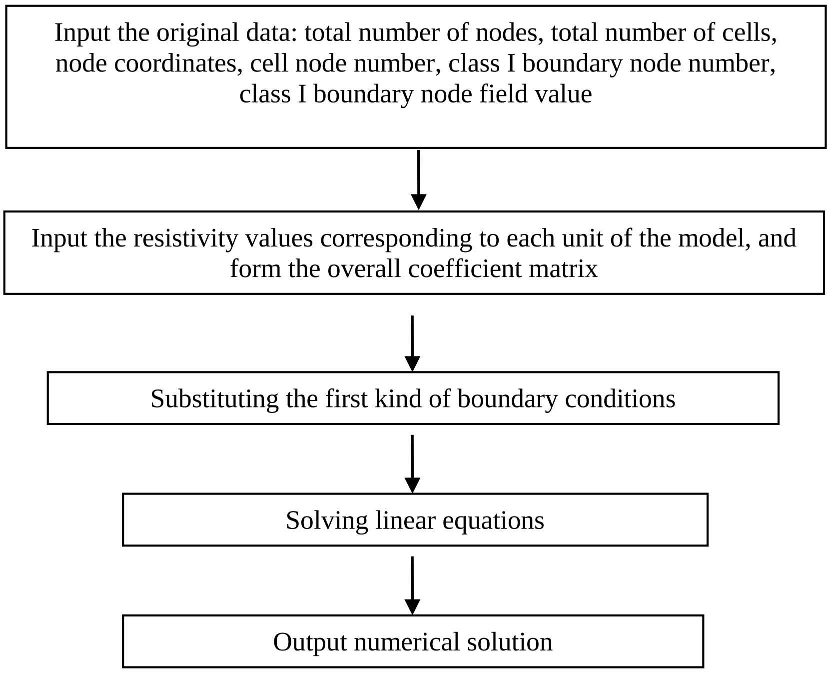

Figure 3. Algorithm flow chart

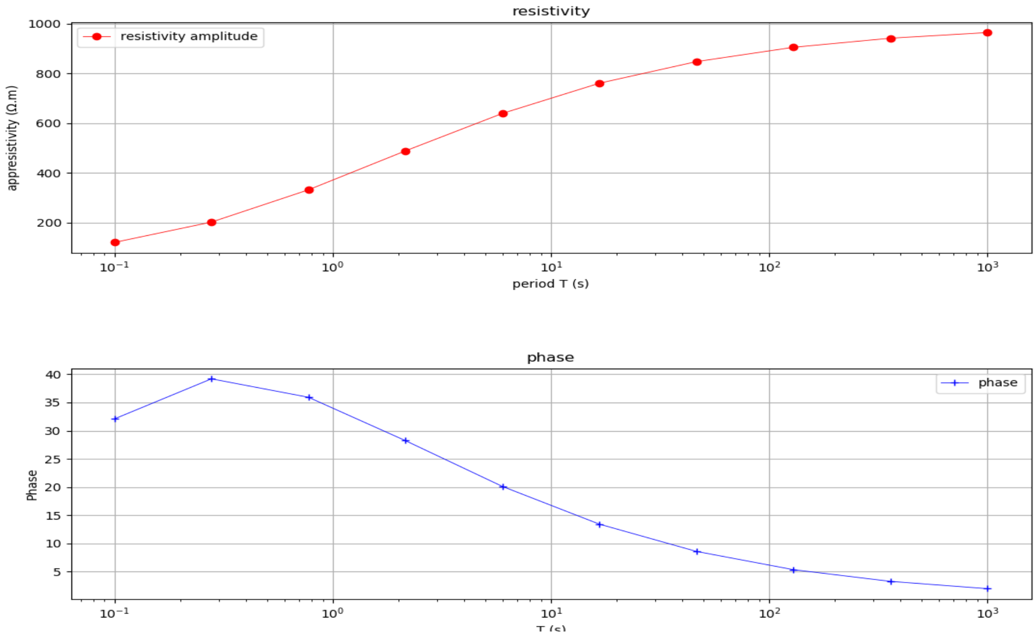

Figure 4. G-type model

Compared with the field value E calculated by the analytical method, the calculated field value E is as follows:

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1	FEM method	analytic method
2	119.6410249	119.641022
3	133.9210041	133.9210064
4	150.9618882	150.9619
5	170.7847225	170.7847503
6	193.3700609	193.3701144
7	218.6461046	218.6461975
8	246.4787904	246.4789421
9	276.6657743	276.6660112
10	308.9356957	308.936053
11	342.9533916	342.9539151
12	378.3308954	378.3316438
13	414.6432206	414.6442669
14	451.4472528	451.4486868
15	488.3016983	488.303628

(a)

1	FEM method	analytic method
2	32.08177656	32.08181624
3	34.94373703	34.94378725
4	37.03386033	37.03392307
5	38.39175319	38.39183075
6	39.0846658	39.08476078
7	39.19284306	39.19295845
8	38.80022681	38.80036598
9	37.98904368	37.98921045
10	36.83689553	36.83709414
11	35.41532277	35.41555797
12	33.78917242	33.78944944
13	32.0163818	32.01670645
14	30.1479762	30.14835487
15	28.22819082	28.22863056

(b)

Figure 5. (a): The results of E which calculated by the finite element method and analytical method; (b): The results of phase which calculated by the finite element method and analytic method

It can be seen from Fig. 5 that the calculation results of finite element method and analytical method are basically the same. After calculation, the relative error rate of apparent resistivity is 0.0037%, and the relative error of phase is 0.0025 °;

4. SUMMARY

By writting and running the finite element forward program, the G-type model of layered earth medium is calculated and the calculation results are analyzed. It is proved that it is feasible to use the finite element method for MT two-dimensional forward modeling, and the finite element method has the advantages of convenient calculation and high accuracy of calculation results.

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