ISSN: 2472-3703 DOI: 10.6911/WSRJ.202003_6(3).0027

Positive Solutions and Green Function of Second-Order Periodic Boundary Value Problems

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Abstract

In this paper, by employing fixed-point index theorem on the cone, we study the existence of positive solutions for second-order periodic boundary value problem (BVP for short): $u''(t)-\alpha u'(t)+\beta u=\lambda f(t,u(t))$, a.e. $t\in[0,2\pi]$; $u(0)=u(2\pi)$, $u'(0)=u'(2\pi)$ **where** $\alpha>0$, $\beta>0$, $\frac{\alpha}{2}>\sqrt{\beta}$ **are constants and** $\lambda>0$ **is a parameter.**

Keywords

Second-order periodic boundary value problems; Positive solutions; Fixed point index theorem.

1. INTRODUCTION

Periodic boundary value problems for differential equations play a very important role in both theory and applications. Recently, the periodic boundary value problems of second-order differential equations with parameters have received much attention [1-18]. We are interested in the existence of positive solutions for the second-order nonlinear periodic boundary value problems with parameters

$$u''(t)-\alpha u'(t)+\beta u=\lambda f(t,u(t)),\ \text{a.e. } t\in[0,2\pi]; \tag{1.1}$$

$$u(0)=u(2\pi),\quad u'(0)=u'(2\pi) \tag{1.2}$$

Where $\alpha>0$, $\beta>0$, $\frac{\alpha}{2}>\sqrt{\beta}$ are constants and $\lambda>0$ is a parameter. Here, by a positive solution we mean a function $u^*(t)$ which is positive on $[0,2\pi]$ and satisfies differential equation (1.1) and the boundary conditions (1.2). It is assumed throughout that

(H_1) $f:[0,2\pi]\times[0,+\infty)\to R^+$ is a continuous function, $\int_0^{2\pi} f(t,u(t))dt<+\infty$,

(H_2) $\forall 0\le u_1\le u_2<+\infty,\ f(t,u_1)\le f(t,u_2)$;

(H_3) $\lim\limits_{u\to+\infty}\frac{f(t,u)}{u}=+\infty$.

2. PRELIMINARY LEMMAS

In this section, we present some preliminary lemmas that will be used in the proofs of the main results.

Lemma 2.1. If $r(t)$ is a solution of the BVP

ISSN: 2472-3703 DOI: 10.6911/WSRJ.202003_6(3).0027

$$\begin{cases} r''(t)-\alpha r'(t)+\beta r(t)=0,\ \ 0\le t\le 2\pi\ ,\alpha>0,\beta>0,\dfrac{\alpha}{2}>\sqrt{\beta} \\ r(0)=r(2\pi), \\ r'(0)-r'(2\pi)=1 \end{cases} \tag{2.1}$$

Then $r(t)$ is given by

$$r(t)=\frac{-(1-e^{2\pi\lambda_2})e^{\lambda_1 t}+(1-e^{2\pi\lambda_1})e^{\lambda_2 t}}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})} \tag{2.2}$$

Where

$\lambda_1=\frac{\alpha}{2}+\frac{\sqrt{\alpha^2-4\beta}}{2}$, and $\lambda_2=\frac{\alpha}{2}-\frac{\sqrt{\alpha^2-4\beta}}{2}$.

Lemma 2.2. If (2.2) is a solution of the BVP(2.1), then the solution of the BVP(1.1),(1.2) is given by

$u(t)=\lambda\int_0^{2\pi}G(t,s)f(s,u(s))ds$,

Where

$$G(t,s)=\begin{cases} r(t-s) & 0\le s\le t\le 2\pi \\ r(2\pi+t-s) & 0\le t<s\le 2\pi \end{cases}.$$

Proof.

$$u(t)=\lambda\int_0^{2\pi}G(t,s)f(s,u(s))ds$$

$$=\lambda\int_0^{t}r(t-s)f(s,u(s))ds+\lambda\int_t^{2\pi}r(2\pi+t-s)f(s,u(s))ds$$

Then

$$u'(t)=\lambda[r(0)-r(2\pi)]+\lambda\int_0^{t}r'(t-s)f(s,u(s))ds+\lambda\int_t^{2\pi}r'(2\pi+t-s)f(s,u(s))ds$$

$$=\lambda\int_0^{t}r'(t-s)f(s,u(s))ds+\lambda\int_t^{2\pi}r'(2\pi+t-s)f(s,u(s))ds$$

$$u''(t)=\lambda\int_0^{t}r''(t-s)f(s,u(s))ds+\lambda\int_t^{2\pi}r''(2\pi+t-s)f(s,u(s))ds+\lambda f(t,u)$$

$$u''(t)-\alpha u'(t)+\beta u(t)=\lambda\int_0^{t}[r''(t-s)-\alpha r'(t-s)+\beta r(t-s)]f(s,u(s))ds$$

$+\lambda\int_t^{2\pi}[r''(2\pi+t-s)-\alpha r'(2\pi+t-s)+\beta r(2\pi+t-s)]f(s,u(s))ds+\lambda f(t,u)$ = $\lambda f(t,u)$,and u(t) meets the following conditions

$$u(0)=\lambda\int_0^{2\pi}r(2\pi-s)f(s,u(s))ds=u(2\pi)$$

$$u'(0)=\lambda\int_0^{2\pi}r'(2\pi-s)f(s,u(s))ds=u'(2\pi).$$

Lemma 2.3. BVP (2.1),(2.2) is equivalent to the following integral equation

$u(t)=\lambda\int_0^{2\pi}G(t,s)f(s,u(s))ds$,

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Where

$$G(t,s)=\begin{cases}\dfrac{(e^{2\pi\lambda_2}-1)e^{\lambda_1(t-s)}+(1-e^{2\pi\lambda_1})e^{\lambda_2(t-s)}}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})} & 0\le s\le t\le 2\pi,\\ \dfrac{(e^{2\pi\lambda_2}-1)e^{\lambda_1(2\pi+t-s)}+(1-e^{2\pi\lambda_1})e^{\lambda_2(2\pi+t-s)}}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})} & 0\le t<s\le 2\pi.\end{cases}$$

$$=\begin{cases}\dfrac{e^{\lambda_2(t-s)}-e^{\lambda_1(t-s)}+e^{2\pi\lambda_1+\lambda_1(t-s)}(e^{2\pi(\lambda_2-\lambda_1)}-e^{(\lambda_2-\lambda_1)(t-s)})}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})} & 0\le s\le t\le 2\pi,\\ \dfrac{e^{\lambda_2(2\pi+t-s)}-e^{\lambda_1(2\pi+t-s)}+e^{2\pi\lambda_1+\lambda_1(2\pi+t-s)}(e^{2\pi(\lambda_2-\lambda_1)}-e^{(\lambda_2-\lambda_1)(2\pi+t-s)})}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})} & 0\le t<s\le 2\pi.\end{cases}$$

Lemma 2.4. For all $t,s\in[0,2\pi]$,we have

$$\frac{(2\pi-t)e^{2\pi\lambda_1}}{(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}\le G(t,s)\le\frac{(1+e^{2\pi\lambda_1})e^{2\pi\lambda_2}}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}.$$

Proof. When $0\le s\le t\le 2\pi$,

$$G(t,s)=\frac{(e^{2\pi\lambda_2}-1)e^{\lambda_1(t-s)}+(1-e^{2\pi\lambda_1})e^{\lambda_2(t-s)}}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

$$=\frac{e^{\lambda_2(t-s)}-e^{\lambda_1(t-s)}+e^{2\pi\lambda_1+\lambda_1(t-s)}(e^{2\pi(\lambda_2-\lambda_1)}-e^{(\lambda_2-\lambda_1)(t-s)})}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

$$\le\frac{e^{2\pi\lambda_2+\lambda_1(t-s)}+e^{\lambda_2(t-s)}}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

$$\le\frac{e^{2\pi\lambda_2+2\pi\lambda_1}+e^{2\pi\lambda_2}}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

$$=\frac{e^{2\pi\lambda_2}(1+e^{2\pi\lambda_1})}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

$$G(t,s)=\frac{(e^{2\pi\lambda_2}-1)e^{\lambda_1(t-s)}+(1-e^{2\pi\lambda_1})e^{\lambda_2(t-s)}}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

$$=\frac{e^{\lambda_2(t-s)}-e^{\lambda_1(t-s)}+e^{2\pi\lambda_1+\lambda_1(t-s)}(e^{2\pi(\lambda_2-\lambda_1)}-e^{(\lambda_2-\lambda_1)(t-s)})}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

$$\ge\frac{e^{2\pi\lambda_1}\left(e^{2\pi(\lambda_2-\lambda_1)}-e^{(\lambda_2-\lambda_1)(t-s)}\right)}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

$$=\frac{e^{2\pi\lambda_1}\left(\sum_{n=0}^{\infty}\frac{(2\pi(\lambda_2-\lambda_1))^n}{n!}-\sum_{n=0}^{\infty}\frac{((\lambda_2-\lambda_1)(t-s))^n}{n!}\right)}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

$$\ge\frac{e^{2\pi\lambda_1}(\lambda_2-\lambda_1)(2\pi-(t-s))}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}=\frac{e^{2\pi\lambda_1}(2\pi-(t-s))}{(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

$$\ge\frac{e^{2\pi\lambda_1}(2\pi-t)}{(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}.$$

ISSN: 2472-3703 DOI: 10.6911/WSRJ.202003_6(3).0027

When $0 \le t < s \le 2\pi$,

$$G(t,s)=\frac{(e^{2\pi\lambda_2}-1)e^{\lambda_1(2\pi+t-s)}+(1-e^{2\pi\lambda_1})e^{\lambda_2(2\pi+t-s)}}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

$$=\frac{e^{\lambda_2(2\pi+t-s)}-e^{\lambda_1(2\pi+t-s)}+e^{2\pi\lambda_1+\lambda_1(2\pi+t-s)}(e^{2\pi(\lambda_2-\lambda_1)}-e^{(\lambda_2-\lambda_1)(2\pi+t-s)})}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

$$\ge\frac{e^{2\pi\lambda_1}\left(e^{2\pi(\lambda_2-\lambda_1)}-e^{(\lambda_2-\lambda_1)(2\pi+t-s)}\right)}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

$$=\frac{e^{2\pi\lambda_1}\left(\sum_{n=0}^{\infty}\frac{(2\pi(\lambda_2-\lambda_1))^n}{n!}-\sum_{n=0}^{\infty}\frac{((\lambda_2-\lambda_1)(2\pi+t-s))^n}{n!}\right)}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

$$\ge\frac{e^{2\pi\lambda_1}(\lambda_2-\lambda_1)(2\pi-(2\pi+t-s))}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

$$=\frac{e^{2\pi\lambda_1}((2\pi-t)-(2\pi-s))}{(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}\ge\frac{e^{2\pi\lambda_1}(2\pi-t)}{(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

$$G(t,s)=\frac{(e^{2\pi\lambda_2}-1)e^{\lambda_1(2\pi+t-s)}+(1-e^{2\pi\lambda_1})e^{\lambda_2(2\pi+t-s)}}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

$$=\frac{e^{\lambda_2(2\pi+t-s)}-e^{\lambda_1(2\pi+t-s)}+e^{2\pi\lambda_1+\lambda_1(2\pi+t-s)}(e^{2\pi(\lambda_2-\lambda_1)}-e^{(\lambda_2-\lambda_1)(2\pi+t-s)})}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

$$\le\frac{e^{2\pi\lambda_1+\lambda_1(2\pi+t-s)}e^{2\pi(\lambda_2-\lambda_1)}+e^{\lambda_2(2\pi+t-s)}}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

$$\le\frac{e^{4\pi\lambda_1}e^{2\pi(\lambda_2-\lambda_1)}+e^{2\pi\lambda_2}}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

$$=\frac{e^{2\pi\lambda_2}(1+e^{2\pi\lambda_1})}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}$$

By Lemma 2.4, For all $t\in\left[\frac{\pi}{4},\frac{7\pi}{4}\right], s\in[0,2\pi]$, we know

$$m=\frac{\pi e^{2\pi\lambda_1}}{4(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}\le G(t,s)\le\frac{(1+e^{2\pi\lambda_1})e^{2\pi\lambda_2}}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})}=M.$$

Where

$$\sigma=\frac{m}{M}=\frac{\pi(\lambda_2-\lambda_1)}{4(1+e^{2\pi\lambda_1})e^{2\pi\lambda_2-\lambda_1}}.$$

Define the cone K by

$$K=\left\{u\in C[0,2\pi];\ \min_{\frac{\pi}{4}\le t\le\frac{7\pi}{4}}u(t)\ge\sigma\|u\|\right\},$$

$$\|u\|=\sup_{t\in[0,2\pi]}\{|u(t)|:u\in C[0,2\pi]\}=\max_{t\in[0,2\pi]}|u(t)|.$$

$K\subset E$ is a closed cone in E.

ISSN: 2472-3703 DOI: 10.6911/WSRJ.202003_6(3).0027

Because K contains the origin, $0 \in K$

$$\forall u_n \in K, u_n \to u \Leftrightarrow u_n(t) \overset{\text{uniform convergence}}{\to} u(t),\ u_n(t) \in K$$

$$u_n(t) \geq \min u_n(t) \geq \sigma \|u_n\|, n \to \infty$$

$$u(t) \geq \sigma \|u(t)\|,\ u(t) \in K$$

$$\forall u, v \in K, \gamma \in [0,1]$$

$$\min_{t \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]} u(t) \geq \sigma \|u\|,\quad \min_{t \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]} v(t) \geq \sigma \|v\|$$

$$[\gamma u + (1-\gamma)v](t) \geq \min_{t \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]} [\gamma u + (1-\gamma)v](t) \geq \gamma\sigma\|u\| + (1-\gamma)\sigma\|v\|$$

$$= \sigma(\|\gamma u\| + \|(1-\gamma)v\|) \geq \sigma\|\gamma u + (1-\gamma)v\|.$$

Lemma 2.5. $\forall u \in K$, $\Phi(K) \subset K$.

Proof. By Lemma2.4,we know $m \leq G(t,s) \leq M$, $\forall u \in K$,

$$\min_{t \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]} (\Phi u)(t) = \min_{t \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]} \lambda \int_0^{2\pi} G(t,s) f(s,u(s))ds$$

$$\geq \sigma\lambda \int_0^{2\pi} G(t,s) f(s,u(s))ds$$

$$\geq \lambda \frac{m}{M} \int_0^{2\pi} M f(s,u(s))ds$$

$$\geq \sigma \max_{t \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]} \int_0^{2\pi} G(t,s) f(s,u(s))ds$$

$$\geq \sigma \|\Phi u\|$$

Therefore

$$\Phi(K) \subset K.$$

Define the mapping,

$$\Phi: C[0,2\pi] \to C[0,2\pi]$$

$$(\Phi u)(t) = \lambda \int_0^{2\pi} G(t,s) f(s,u(s))ds.$$

Lemma 2.6. $\Phi: C[0,2\pi] \to C[0,2\pi]$ is a completely continuous operator.

Lemma 2.7. Let E be a Banach space and $K \subset E$ a closed cone in E.

$K_r = \{x \in K; \|x\| < r\}$ and $\partial K_r = \{x \in K; \|x\| = r\}$,

Let $\Phi: \bar{K}_r \to K$ be a completely continuous operator.

(i) $\|x\| \leq \|\Phi x\|, \forall x \in \partial K_r$, then $i(\Phi, K_r, K) = 0$;

(ii) $\|x\| \geq \|\Phi x\|, \forall x \in \partial K_r$, then $i(\Phi, K_r, K) = 1$.

Lemma 2.8. Let E be a Banach space and $K \subset E$ a closed cone in E.

Assume Ω is open subsets of E with $0 \in \Omega$; Let $\Phi: \overline{\Omega} \to K$ be a completely continuous operator. $\Phi x \neq \mu x, x \in K \cap \partial\Omega$, $\mu \geq 1$, then $i(\Phi, K \cap \Omega, K) = 1$.

ISSN: 2472-3703 DOI: 10.6911/WSRJ.202003_6(3).0027

3. MAIN RESULTS

In this section, we discuss the existence of a positive solution of BVP (1.1), (1.2).

Theorem 3.1. If hypothesis(H1),(H2),(H3) are true, The BVP (1.1), (1.2) has at least one positive solution when a is sufficiently small; and has no positive solution when a is sufficiently large.

Proof. For $\forall r_1 > 0$,

Let $\sigma_1 = \dfrac{r_1}{M \max\limits_{u\in\partial r_1} \int_0^{2\pi} f(s,u(s))ds}$, $K_{r_1} = \{u \in K; \|u\| < r_1\}$,

When $\lambda < \sigma_1$, for $\forall u \in \partial K_{r_1}$.

We have

$$\|\Phi u\| = \max_{t\in[0,2\pi]} \lambda \int_0^{2\pi} G(t,s) f(s,u(s))ds \le M\lambda \max_{u\in\partial k_{r_1}} \int_0^{2\pi} G(t,s) f(s,u(s))ds$$

$$< M\sigma_1 \max_{u\in\partial k_{r_1}} \int_0^{2\pi} f(s,u(s))ds = r_1 = \|u\|.$$

By Lemma2.7, know $i(\Phi, K_{r_1}, K) = 1$.

For $\lim\limits_{u\to+\infty} \dfrac{f(t,u)}{u} = +\infty$, know $\exists N > 0$, when $u > N$, $f(t,u(t)) \ge \eta u$, and $\eta > \dfrac{3}{2\pi m\lambda}$.

Let $r_2 = N/\sigma$, $K_{r_2} = \{u \in K; \|u\| < r_2\}$, for $\forall u \in \partial K_{r_2}$

$$\min_{t\in\left[\frac{\pi}{4},\frac{7\pi}{4}\right]} u(t) \ge \sigma\|u\| = \sigma r_2 = N.$$

Therefore

$$\max_{t\in[0,2\pi]} (\Phi u)(t) = \lambda \max_{t\in[0,2\pi]} \left| \int_0^{2\pi} G(t,s) f(s,u(s))ds \right|$$

$$\ge \lambda \min_{t\in\left[\frac{\pi}{4},\frac{7\pi}{4}\right]} \int_{\frac{\pi}{4}}^{\frac{7\pi}{4}} G(t,s) f(s,u(s))ds$$

$$\ge \lambda m \int_{\frac{\pi}{4}}^{\frac{7\pi}{4}} f(s,u(s))ds \ge \lambda m\eta \int_{\frac{\pi}{4}}^{\frac{7\pi}{4}} u(s)ds$$

$$\ge \lambda m\eta \int_{\frac{\pi}{4}}^{\frac{7\pi}{4}} \|u(s)\|ds = \frac{3\pi}{2}\lambda m\eta\|u\| > \|u\|$$

$$> M\sigma_1 \max_{u\in\partial k_{r_1}} \int_0^{2\pi} f(s,u(s))ds = r_1 = \|u\|,$$

$\|\Phi u\| \ge \|u\|$, By Lemma 2.7, $i(\Phi, K_{r_2}, K) = 0$.

$r_1 < r_2$, We know by the additivity of the fixed point exponent

$$i(\Phi, K_{r_2} \setminus \overline{K}_{r_1}, K) = i(\Phi, K_{r_2}, K) - i(\Phi, K_{r_1}, K) = -1.$$

The Φ has at least one fixed point u in $K_{r_2} \setminus \overline{K}_{r_1}$, and $\Phi u = u$.

$$u(t) = \lambda \int_0^{2\pi} G(t,s) f(s,u(s))ds,$$

ISSN: 2472-3703 DOI: 10.6911/WSRJ.202003_6(3).0027

Thus u(t) is a positive solution of the boundary value problem (1.1) , (1.2).

For $\lim_{u\to+\infty}\frac{f(t,u)}{u}=+\infty$, know $\exists c>0,$ when $u>\eta$, $f(t,u(t))\geq cu$ and $u\in K$,

(i) $\|u\|\geq\frac{\eta}{\sigma}$, $\min_{t\in\left[\frac{\pi}{4},\frac{7\pi}{4}\right]}u(t)\geq\sigma\|u\|\geq\eta$,

$$\|u\|=\sup_{t\in[0,2\pi]}\{|u(t)|:u\in C[0,2\pi]\}=\max_{t\in[0,2\pi]}|u(t)|$$

$$\geq\lambda\min_{t\in\left[\frac{\pi}{4},\frac{7\pi}{4}\right]}\left|\int_0^{2\pi}G(t,s)f(s,u(s))ds\right|$$

$$\geq\lambda\min_{t\in\left[\frac{\pi}{4},\frac{7\pi}{4}\right]}\int_{\frac{\pi}{4}}^{\frac{7\pi}{4}}G(t,s)f(s,u(s))ds$$

$$\geq\lambda mc\int_{\frac{\pi}{4}}^{\frac{7\pi}{4}}u(s)ds\geq\frac{3}{2}\lambda m\pi c\sigma\|u\|,$$

So $\lambda\leq\frac{2}{3m\pi c\sigma}$, it is in contradiction with knowing that λ is sufficiently large.

(ii) If $\|u\|<\frac{\eta}{\sigma}$,

$$\frac{\eta}{\sigma}>\|u\|=\sup_{t\in[0,2\pi]}\{|u(t)|:u\in C[0,2\pi]\}=\max_{t\in[0,2\pi]}|u(t)|$$

$$=\lambda\max_{t\in[0,2\pi]}\left|\int_0^{2\pi}G(t,s)f(s,u(s))ds\right|$$

$$\geq\lambda\min_{t\in\left[\frac{\pi}{4},\frac{7\pi}{4}\right]}\left|\int_0^{2\pi}G(t,s)f(s,u(s))ds\right|$$

$$\geq\lambda\min_{t\in\left[\frac{\pi}{4},\frac{7\pi}{4}\right]}\int_0^{2\pi}G(t,s)f(s,u(s))ds$$

$$\geq\lambda m\int_0^{2\pi}f(s,0)ds.$$

So $\lambda\leq\frac{\eta}{\sigma m\int_0^{2\pi}f(s,0)ds}$,it is in contradiction with knowing that λ is sufficiently large.

Let $\Lambda=\{\lambda>0,\text{The BVP (1.1), (1.2) has at least one positive solution}\}$, the first part of theorem 3.1 proves that $\lambda_0\in\Lambda$,

When $\lambda\leq\lambda_0$, $\lambda\in\Lambda$, $(0,\lambda_0]\subset\Lambda$, let $\lambda^*=\sup\Lambda$, we obtain

Theorem 3.2. If hypothesis(H1),(H2),(H3) are true, there are $\lambda^*\in\Lambda$, thus $(0,\lambda^*]\subset\Lambda$.

Proof. Let $\{\lambda_n\}_{n=1}^{\infty}$ is a monotone increasing sequence, and $\lim_{n\to\infty}\lambda_n=\lambda^*$.

For $\lim_{u\to+\infty}\frac{f(t,u)}{u}=+\infty$, know $\exists N>0$,when $u>N$, $f(t,u(t))\geq\eta u$, $2\pi m\lambda_n\eta>1$.

If $\left\|u_{\lambda_n}\right\| > \frac{N}{\sigma}$,

$$\min_{\frac{\pi}{4}\le t\le\frac{7\pi}{4}} u_{\lambda_n}(t) \ge \sigma\left\|u_{\lambda_n}\right\|$$

Therefore

$$\left\|u_{\lambda_n}\right\| \ge \lambda_n \min_{\frac{\pi}{4}\le t\le\frac{7\pi}{4}} \int_0^{2\pi} u_{\lambda_n}(s)ds \ge \lambda_n m\eta \ge 2\pi\lambda_n m\eta\sigma\left\|u_{\lambda_n}(t)\right\| > \left\|u_{\lambda_n}\right\|.$$

So there are constants $0 < L < +\infty$, $\left\|u_{\lambda_n}\right\| \le L$,for all n. $\{u_{\lambda_n}\}_{n=1}^{\infty}$ is uniformly bounded.

$$\left\|u'_{\lambda_n}\right\| = \max_{t\in[0,2\pi]} \lambda_n \int_0^{2\pi} \frac{\partial G(t,s)}{\partial t} f(s,u(s))ds$$

$$\le \lambda_n \frac{(\lambda_2+\lambda_1 e^{2\pi\lambda_1})e^{2\pi\lambda_2}}{(\lambda_2-\lambda_1)(1-e^{2\pi\lambda_2})(1-e^{2\pi\lambda_1})} \int_0^{2\pi} f(s,u(s))ds = Q,$$

thus $\left\|u_{\lambda_n}(t_1) - u_{\lambda_n}(t_2)\right\| \le \left|\int_{t_1}^{t_2} \left\|u'_{\lambda_n}\right\| dt\right|$, $\{u_{\lambda_n}\}_{n=1}^{\infty}$ is equicontinuity.

By theorem $Ascoli - Arzela$, $\{u_{\lambda_n}\}_{n=1}^{\infty}$

Have uniformly convergent subsequences $\{u_{\lambda_n}\}_{n=1}^{\infty}$, $\lim_{n\to\infty} u_{\lambda_n}(t) \overset{\text{uniform convergene}}{=} u^*(t)$.

$$u_{\lambda_n}(t) = \lambda_n \int_0^{2\pi} G(t,s) f(s,u_{\lambda_n}(s))ds,$$

By Lebesgue dominated convergence theorem, let $n \to \infty$ we have $\lambda^* \int_0^{2\pi} G(t,s) f(s,u^*(s))ds$,

$u^*(t)$ is a positive solution of the boundary value problem (1.1), (1.2), and $\lambda^* \in \Lambda$.

The proof is complete.

ACKNOWLEDGMENTS

This paper was supported by

1. National innovation and entrepreneurship project for College Students: Study on soybean potential yield in Qigihar and Harbin regions of Heilongjiang (No.201810223017),

2. Textbook construction research project of the China agricultural science and education fund: Research and practice on the reform of process assessment model in linear algebra (No.NKJ201802045).

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