Research on Hochschild Cohomology of Commutative Coalgebras

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Abstract

The Hochschild cohomology of commutative coalgebra is discussed with reference to the research method of Hochschild cohomology in algebra. Using the definitions and calculation methods of Hochschild cohomology given by Y. DOi, the zero-order Hochschild cohomology group of bi-comodule on coalgebra is studied, we obtain the zero-order Hochschild cohomology group of its coalgebra.

Keywords

Commutative coalgebra; Bi-comodule; Hochschild cohomology group.

1. INTRODUCTION

Since the concept and theory of categories were put forward by MacLane and Eilenberg in 1945, it has been used in many branches of mathematics, such as algebraic geometry, topology, differential geometry and function theory [1,2]. Homology group theory is an important branch of algebra research, it mainly studies the structure of finite dimensional algebra, the indecomposable representation and the construction of module categories. Coalgebra is defined by the dual of algebra, on the basis of the study of the module category of algebra, the study of the comodule category of coalgebra has important significance for discussing the structure and representation of coalgebra [3]. The calculation of the Hochschild homology group of coalgebra is more difficult. Therefore, the calculation of zero-order Hochschild cohomology group of two coalgebras is given below.

2. PREPARATIVE KNOWLEDGE

2.1. Coalgebra and Bi-comodule

Definition 1[4] A $A-$coring is a coalgebra in category $\left({}_{A}M_{A},\otimes_{A},A\right)$.Specifically, a $A-$coring C is a $\left(A,A\right)-$bi-module with $\left(A,A\right)-$bi-module mapping $\Delta_{C}:C\to C\otimes_{A}C$, which Makes the following condition true:

$$\left(\Delta_{C}\otimes_{A}id_{C}\right)\circ\Delta_{C}=\left(id_{C}\otimes_{A}\Delta_{C}\right)\circ\Delta_{C}$$

Where, id_{C} is the identity mapping on C, Δ_{C} is called coproduct.

Definition 2[5] Assume M is a $\left(A,A\right)-$ bi-module, M is a right $C-$ comodule with right coaction $\rho^{+}:M\to M\otimes_{A}C$, and M is a left $C-$comodule with left coaction $\rho^{-}:M\to C\otimes_{A}M$. If M satisfies $\left(I_{C}\otimes\rho^{+}\right)\circ\rho^{-}=\left(\rho^{-}\otimes I_{C}\right)\circ\rho^{+}$, then it is a $\left(C,C\right)-$bi-module.

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2.2. Hochschild Cohomology Group

Definition 3 [6] For each integer $n \geq -1$, let $S^n(C)$ represent $(n+2)$-th tensor products of C. If we define the left and right comodule structure over $S^n(C)$ as follows:

$$\rho^-(c_0 \otimes c_1 \otimes \cdots \otimes c_{n+1}) = \Delta(c_0) \otimes c_1 \otimes \cdots \otimes c_{n+1}$$

$$\rho^+(c_0 \otimes c_1 \otimes \cdots \otimes c_{n+1}) = c_0 \otimes c_1 \otimes \cdots \otimes c_n \otimes \Delta(c_{n+1})$$

Then $S^n(C)$ becomes a $(C,C)-$ bi-module. Obviously, as a left C^e- comodule, $S^n(C)$ is injective. For each $n \geq 0$, by

$$d^n(c_0 \otimes c_1 \otimes \cdots \otimes c_{n+1}) = \sum_{i=0}^{n+1}(-1)^i c_0 \otimes \cdots \otimes \Delta(c_i) \otimes \cdots \otimes c_{n+1}$$

We define a mapping $d^n : S^n(C) \to S^{n+1}(C)$,and for each $n \geq 1$, by

$$s^n(c_0 \otimes c_1 \otimes \cdots \otimes c_{n+1}) = \varepsilon(c_0)c_1 \otimes \cdots \otimes c_{n+1}$$

We define a right C comodule mapping $s^n : S^n(C) \to S^{n-1}(C)$.

It is easy to verify that $s^{n+1}d^n + d^{n-1}s^n = I(n \geq 1)$ .This suggests that $C = S^{-1}(C) \xrightarrow{\Delta} S^0(C) \xrightarrow{d^0} S^1 \xrightarrow{d^1} \cdots$ is the injective decomposition of C as the left C^e- comodule. we see $S^0(C) \cong C \otimes C$ and $C^e = C \otimes C^{op}$ are compatible as the right C^e-comodules. More generally, we have $S^n(C) \cong C^e \otimes C^{[n]}$ as a C^e-comodule. For each $n > 0$, $C^{[n]}$ is the n-tensor product of C ,and $C^{[0]} = k$, its differential $D^n : N \otimes C^{[n]} \to N \otimes C^{[n+1]}$ is defined as follows:

$$D^n(v \otimes c_1 \otimes \cdots \otimes c_n) = \rho^+(v) \otimes c_1 \otimes \cdots c_n + \sum_{i=0}^{n}(-1)^i v \otimes c_1 \otimes \cdots \otimes \Delta(c_i) \otimes \cdots \otimes c_n$$

$$+(-1)^{n+1}\sum v_{(0)} \otimes c_1 \otimes \cdots \otimes c_n \otimes v_{(-1)}$$

Denote $\rho^-(v) = \sum v_{(-1)} \otimes v_{(0)} \in C \otimes N$, we obtain the zero-order Hochschild cohomology group, called $Hoch^0$, then

$$Hoch^0(N,C) = \{\alpha \in N \mid t\rho^-(\alpha) = \rho^+(\alpha)\}.$$

3. HOCHSCHILD COHOMOLOGY GROUP OF TWO COMMUTATIVE COALGEBRAS

3.1. Definition

Definition 1[7] Assume $C = K[x]$ is a vector space over field K ,define $\Delta : C \to C \otimes C$ which makes $x^n \mapsto \sum_{i+j=n} x^i \otimes x^j$, it can be verified that Δ satisfies the associative law, then (C,Δ) is a $K-$coalgebra. we call (C,Δ) polynomial coalgebra.

Definition 2[8] Assume C is a vector space based on s,c over field K ,define coproduction $\Delta : C \to C \otimes C$ which makes

$$\Delta(s) = s \otimes c + c \otimes s, \quad \Delta(c) = c \otimes c - s \otimes s$$

It can be verified that Δ satisfies conditions of coassociative law, then (C,Δ) is a $K-$ coalgebra. we call (C,Δ) triangular geometry coalgebra.

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3.2. Theorem

Theorem1 Assume $C=K[x]$ is a polynomial coalgebra over field K, V is a linear space, the linear mappings T_1,T_2 are locally nilpotent, define linear mappings $\rho^+:V\to V\otimes C$, $\rho^-:V\to C\otimes V$ which make

$$\rho^+(v)=\sum_{k\geq 0}T_1^kv\otimes x^k,\ \rho^-(v)=\sum_{k\geq 0}x^k\otimes T_2^kv, \text{ and } T_1T_2=T_2T_1$$

Then

$$Hoch^0(V,C)=Ker(T_2-T_1)$$

Proof According to the known conditions, $\rho^+:V\to V\otimes C$, $\rho^-:V\to C\otimes V$ make the following two formulas hold: $\rho^+(v)=\sum_{k\geq 0}T_1^kv\otimes x^k$, $\rho^-(v)=\sum_{k\geq 0}x^k\otimes T_2^kv$.

It's easy to verify that $(I\otimes\Delta)\rho^+=(\rho^+\otimes I)\rho^+$ holds, so V is a right $C-$ comodule. similarly ,we can prove that $(\Delta\otimes I)\rho^-=(I\otimes\rho^-)\rho^-$ holds, so V is a left $C-$comodule. In the following we verify $(I\otimes\rho^+)\rho^-=(\rho^-\otimes I)\rho^+$, i.e.

$$(I\otimes\rho^+)\rho^-(v)=(\rho^-\otimes I)\rho^+(v)$$

$$\Rightarrow\sum_{i\geq 0}\sum_{j\geq 0}x^i\otimes T_1^iT_2^jv\otimes x^j=\sum_{i\geq 0}\sum_{j\geq 0}x^i\otimes T_2^jT_1^iv\otimes x^j$$

$$\Rightarrow T_1^iT_2^j=T_2^jT_1^i\ (\forall i,j\geq 0)$$

$$\Rightarrow T_1T_2=T_2T_1$$

So (V,ρ^+,ρ^-) is a (C,C)- bi-module. According to the definition of Hochschild homology, $Hoch^0(V,C)=KerD^0$, and because

$$D^0(v)=\rho^+(v)-\sum v_{(0)}\otimes v_{(-1)}=\rho^+(v)-t\sum v_{(-1)}\otimes v_{(0)}=\rho^+(v)-t\rho^-(v),$$

Thus

$$KerD^0=\{v\in V\mid t\rho^-(v)=\rho^+(v)\}$$

Where t represents the twist of tensor product, that is to say, $t(x\otimes y)=y\otimes x$.

By $\rho^-(v)=\sum_{k\geq 0}x^k\otimes T_2^kv$ knowable, $t\rho^-(v)=t\sum_{k\geq 0}x^k\otimes T_2^kv=\sum_{k\geq 0}T_2^kv\otimes x^k$,and

Because $\rho^+(v)=\sum_{k\geq 0}T_1^kv\otimes x^k$,thus

$$t\rho^-(v)=\rho^+(v)\Leftrightarrow\sum_{k\geq 0}T_2^kv\otimes x^k=\sum_{k\geq 0}T_1^kv\otimes x^k$$

$$\Leftrightarrow T_2^kv=T_1^kv\Leftrightarrow T_2v=T_1v\Leftrightarrow T_2v-T_1v=0$$

$$\Leftrightarrow(T_2-T_1)v=0\Leftrightarrow v\in Ker(T_2-T_1)$$

So

$$Hoch^0(V,C)=KerD^0=Ker(T_2-T_1)$$

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Theorem2 Assume C is a triangular geometry coalgebra over field K ,V is a vector space over field K , Assume $T_{1s},T_{1c},T_{2s},T_{2c}$ are linear mapping of V to V ,and they are interchangeable, simultaneously they satisfy

$$T_{1s}^{\ 2}=T_{1s},T_{1s}=-T_{1c}^{\ 2},T_{1s}\cdot T_{1c}=T_{1c}=T_{1c}\cdot T_{1s}$$

$$T_{2c}^{\ 2}=T_{2c},T_{2c}=-T_{2s}^{\ 2},T_{2s}\cdot T_{2c}=T_{2s}=T_{2c}\cdot T_{2s}$$

Define $\rho^{+}:V\to V\otimes C$ and $\rho^{-}:V\to C\otimes V$ which make

$$\rho^{+}(\boldsymbol{v})=T_{1s}(\boldsymbol{v})\otimes\boldsymbol{c}+T_{1c}(\boldsymbol{v})\otimes\boldsymbol{s},\ \rho^{-}(\boldsymbol{v})=\boldsymbol{c}\otimes T_{2c}(\boldsymbol{v})+\boldsymbol{s}\otimes T_{2s}(\boldsymbol{v}),\ \forall\boldsymbol{v}\in V$$

Then the zero-order Hochschild homology group is $Hoch^{0}(V,C)=Ker(T_{2s}-T_{1c})$.

4. APPLICATION EXAMPLES

Example 1 Assume C is triangular geometry coalgebra over field K ,V is vector space over field K , Assume $V=V_1\oplus V_2\oplus V_3$, T_1 is $V\to V$ linear transformation which makes $T_1(\boldsymbol{v})=\begin{pmatrix}\boldsymbol{0}&\boldsymbol{A}&\boldsymbol{0}\\\boldsymbol{A}&\boldsymbol{0}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\end{pmatrix}\begin{pmatrix}\boldsymbol{\alpha}\\\boldsymbol{\beta}\\\boldsymbol{\gamma}\end{pmatrix}$, $\boldsymbol{\alpha}=\begin{pmatrix}\alpha_{11}\\\alpha_{21}\\\vdots\\\alpha_{k1}\end{pmatrix}$, $\boldsymbol{\beta}=\begin{pmatrix}\beta_{11}\\\beta_{21}\\\vdots\\\beta_{k1}\end{pmatrix}$, $\boldsymbol{v}=\begin{pmatrix}\boldsymbol{\alpha}\\\boldsymbol{\beta}\\\boldsymbol{\gamma}\end{pmatrix}$, T_2 is $V\to V$ linear transformation

Which makes $T_2(\boldsymbol{v})=\begin{pmatrix}\boldsymbol{0}&\boldsymbol{B}&\boldsymbol{0}\\\boldsymbol{B}&\boldsymbol{0}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\end{pmatrix}\begin{pmatrix}\boldsymbol{\alpha}\\\boldsymbol{\beta}\\\boldsymbol{\gamma}\end{pmatrix}$, $\boldsymbol{v}=\begin{pmatrix}\boldsymbol{\alpha}\\\boldsymbol{\beta}\\\boldsymbol{\gamma}\end{pmatrix}$, $\boldsymbol{\alpha}\in V_1,\boldsymbol{\beta}\in V_2,\boldsymbol{\gamma}\in V_3$ with

$$\boldsymbol{A}=\begin{pmatrix}0&0&\cdots&0&i\\0&0&\cdots&i&0\\\vdots&\vdots&\iddots&\vdots&\vdots\\0&i&\cdots&0&0\\i&0&\cdots&0&0\end{pmatrix}_{k\times k},\quad \boldsymbol{B}=\begin{pmatrix}0&0&\cdots&0&-i\\0&0&\cdots&-i&0\\\vdots&\vdots&\iddots&\vdots&\vdots\\0&-i&\cdots&0&0\\-i&0&\cdots&0&0\end{pmatrix}_{k\times k}$$

Then $T_1\cdot T_2=T_2\cdot T_1$, $T_i^3+T_i=0,(i=1,2)$ .Define separately $\rho^{+}:V\to V\otimes C$, $\rho^{-}:V\to C\otimes V$ which make

$$\rho^{+}(\boldsymbol{v})=T_1(\boldsymbol{v})\otimes\boldsymbol{s}-T_1^{2}(\boldsymbol{v})\otimes\boldsymbol{c},\rho^{-}(\boldsymbol{v})=\boldsymbol{s}\otimes T_2(\boldsymbol{v})-\boldsymbol{c}\otimes T_2^{\ 2}(\boldsymbol{v}),\forall\boldsymbol{v}\in V$$

Then

$$Hoch^{0}(V,C)=\left\{\boldsymbol{v}=\begin{pmatrix}\boldsymbol{\alpha}\\\boldsymbol{\beta}\\\boldsymbol{\gamma}\end{pmatrix}\middle|\ \boldsymbol{\alpha},\boldsymbol{\beta}\in Ker\left(\begin{pmatrix}0&0&\cdots&0&2i\\0&0&\cdots&2i&0\\\vdots&\vdots&\iddots&\vdots&\vdots\\0&2i&\cdots&0&0\\2i&0&\cdots&0&0\end{pmatrix}_{k\times k}\right),\ \boldsymbol{\gamma}\text{ is any column vector}\right\}$$

Example 2 Assume $C=K[x]$ is a polynomial coalgebra over real field R ,$V=R[x,y]$, the linear mapping $T_1,T_2:V\to V$ are locally nilpotent, and $T_1T_2=T_2T_1$ .Assume $f\in V$, define linear mappings $\rho^{+}:V\to V\otimes C$, $\rho^{-}:V\to C\otimes V$ which make

$$\rho^{+}(f)=\sum_{k\geq 0}T_1^k f\otimes x^k,\quad \rho^{-}(f)=\sum_{k\geq 0}x^k\otimes T_2^k f$$

Then

$$Hoch^0(V,C)=\left\{f\in V\mid f(x,y)=\sum_{n\geq 0}a_n(x+y)^n, a_n\in R,\forall n\geq 0\right\}$$

Proof According to theorem1, we obtain $Hoch^0(V,C)=Ker(T_2-T_1)$, so

$$Hoch^0(V,C)=\left\{f\in V\mid \frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}\right\}$$

By $f(x,y)\in V=R[x,y]$, let $f(x,y)=\sum_{i\geq 0}\sum_{j\geq 0}a_{ij}x^i y^j, a_{ij}\in R$, we can get the partial derivatives of x and y:

$$\frac{\partial f}{\partial x}=\sum_{i\geq 1}\sum_{j\geq 0}i\cdot a_{ij}x^{i-1}y^j,\quad \frac{\partial f}{\partial y}=\sum_{i\geq 0}\sum_{j\geq 1}j\cdot a_{ij}x^i y^{j-1}$$

According to $\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}$,we obtain $\sum_{i\geq 0}\sum_{j\geq 1}j\cdot a_{ij}x^i y^{j-1}=\sum_{i\geq 1}\sum_{j\geq 0}i\cdot a_{ij}x^{i-1}y^j$ .Compare the coefficients at both ends , we have $a_{10}=a_{01}, a_{20}=\frac{a_{11}}{2}=a_{02}, a_{30}=\frac{a_{21}}{3}=\frac{a_{12}}{3}=a_{03}, a_{40}=\frac{a_{31}}{4}=\frac{a_{22}}{6}=\frac{a_{13}}{4}=a_{04},\cdots$

so $f(x,y)=a_0+a_1(x+y)+\cdots+a_i(x+y)^i+\cdots$ ($a_i\in R, i\geq 0$), thereby

$$Hoch^0(V,C)=\left\{f\in V\mid f(x,y)=\sum_{n\geq 0}a_n(x+y)^n, a_n\in R\right\}$$

5. CONCLUSION

In this paper, firstly we introduce the basic concepts of two coalgebras and the definition of Hochschild cohomology group. Secondly, the calculation of the zero-order Hochschild cohomology groups of polynomial coalgebra and triangular geometry coalgebra are studied. Finally, we apply the conclusions of the theorems to specific examples. This paper is instructive for continuing to study the Hochschild cohomology group of other coalgebras.

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