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Investigation of The Double Pendulum Small Angle Approximation Model

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Abstract

The double pendulum system is a classic mechanical structure that is formed by one pendulum attached to another pendulum. It is a complex nonlinear system, which contains a multi-degree of freedom and the involvement of multi-variables representation. In this essay, the system will be plotted into generalized coordinates and used Lagrangian mechanics and Euler's equation to derive the motion of the equation. A small angle approximation will then be used on the equation of motion to transform the chaotic system into a simple linear system. This will enable the direct numerical calculation of the function.

Keywords

Double pendulum; Small angle approximation; Normal modes.

1. REASON FOR USING LAGRANGIAN MECHANICS

The double pendulum consists of two identical pendulum bobs G_1 and G_2, with an identical mass m connected by two same weightless rods L_1 and L_2 with the length of $\boldsymbol{l}$. Putting this system into a Cartesian coordinate with the origin at the point where a double pendulum is connected to the ceiling. (x-axis placed horizontally and y-axis placed parallel to the gravity). The position of the two-point G_1 and G_2 of the two-point masses can all be expressed with their components (x_1, y_1, z_1). Newtonian mechanics is the investigation of forces, vectors, and momentum. Therefore, the constraints of the system must be considered. In this system, there are four constraints: the oscillation of the system only happening in the xy plane and the geometric constraints of both rigid rods. The constraint equation can then be expressed as follows:

$$z_1 = 0$$

$$z_2 = 0$$

$$x_1^{\,2} + y_1^{\,2} = l^2 \tag{1}$$

$$(x_2^{\,2} + y_2^{\,2}) - l^2 = l^2$$

$$x_2^{\,2} + y_2^{\,2} = 2l^2 \tag{2}$$

With these two constraints function (1) and (2), an additional four Newtonian component functions will be needed for the derive of the general equation of motion of this system

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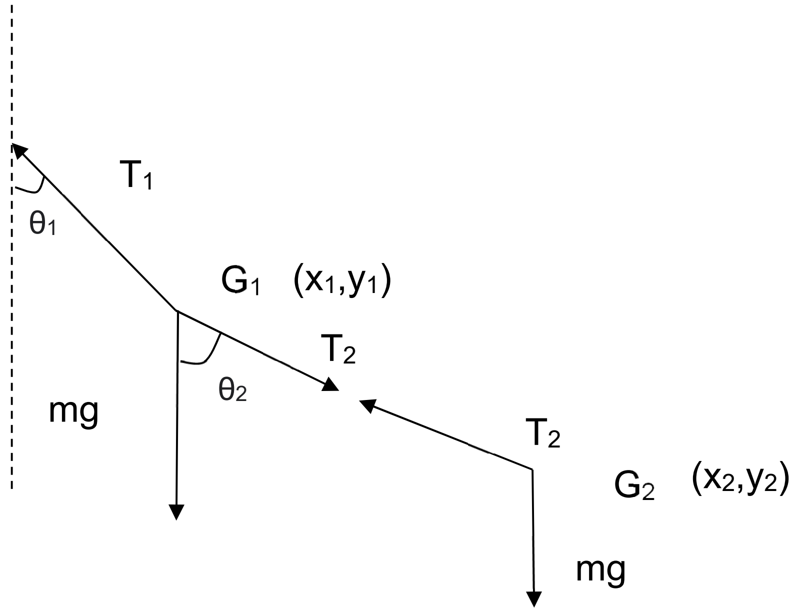

Figure 2. The free body diagram of the double pendulum system

Using the free body diagram, the component newton’s equation are

$$\ddot{x_1} = \frac{d{x_1}^2}{dt^2}$$

$$\ddot{y_1} = \frac{d{y_1}^2}{dt^2}$$

$$-T_1 \frac{x_1}{\sqrt{{x_1}^2+{y_1}^2}} + T_2 \frac{x_2}{\sqrt{{x_2}^2+{y_2}^2}} = m\ddot{x_1} \tag{3}$$

$$T_1 \frac{y_1}{\sqrt{{x_1}^2+{y_1}^2}} - T_2 \frac{y_2}{\sqrt{{x_2}^2+{y_2}^2}} - mg = m\ddot{y_1} \tag{4}$$

$$-T_2 \frac{x_2}{\sqrt{{x_2}^2+{y_2}^2}} = m\ddot{x_2} \tag{5}$$

$$T_2 \frac{y_2}{\sqrt{{x_2}^2+{y_2}^2}} - mg = m\ddot{y_2} \tag{6}$$

These four component equations combined with the two constraint functions are what the equation of motion derived from. There are a total of 6 unknown variables with 6 simultaneous equations, which is solvable from a theoretic point of view. The reason that makes this point mass system become complicated is that the constraint force of the rope and the constraint function has made Newton's component forms become dependent.

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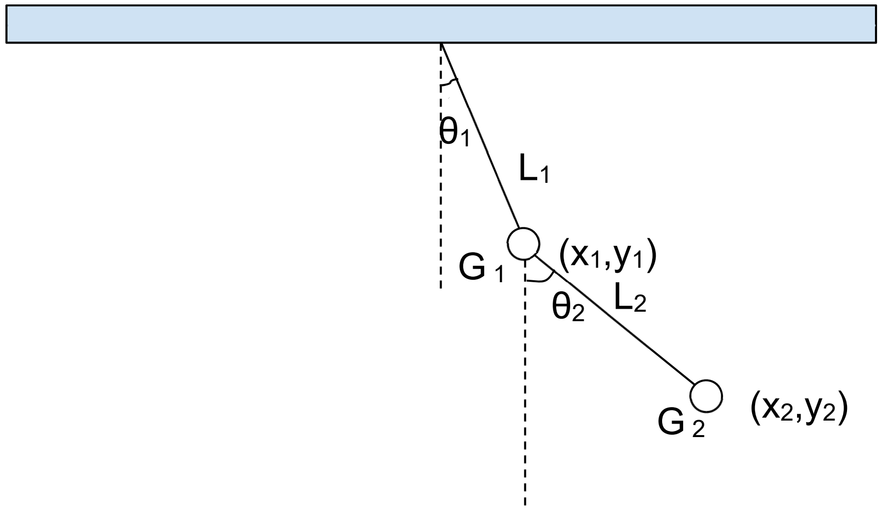

Figure 3. The set up of the double pendulum system in a Polar Coordinate

The use of generalized coordinates can reduce some effort in this case. Generalized coordinate means the system must be independent and holonomic. For instance, the double pendulum can be plotted into polar coordinates(figure 3), where the motion of the points G_1 and G_2 will be determined by the two angles θ_1 and θ_2. This can effectively avoid the problem of tension in the system since there is no relationship between the angles and the tension force.

Lagrangian is the most effective way of dealing with the holonomic system, it uses the kinetic and potential energy of the system to set up the motion of the equation, without considering the acceleration and angular acceleration.

2. EQUATION OF MOTION (EOM)

The Lagrange equation is

$$\boldsymbol{L} = \boldsymbol{T} - \boldsymbol{V} \tag{7}$$

The kinetic energy T can be first determined.

Particle one, G_1 is constrained to move in an arc of a circle with a radius of l , so it is just doing circular motion and the tangential velocity for a circular motion is the angular velocity $\dot{\theta}$, the derivative of angular displacement with respect to time, multiply the radius of the circle. Therefore the kinetic energy of this point one is

$$\tfrac{1}{2}m(l\dot{\theta}_1)^2 \tag{8}$$

The kinetic energy of particle two, G_2 is more complicated since it is no longer constrained to an arc of a circle, so the cartesian form will first be used to represent the kinetic energy of the second particle. (coordinate refer back to figure 1)

$$\frac{1}{2}m(\dot{x_2}^2 + \dot{y_2}^2)$$

$$T = \tfrac{1}{2}m(l\dot{\theta}_1)^2 + \tfrac{1}{2}m(\dot{x_2}^2 + \dot{y_2}^2) \tag{9}$$

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The cartesian coordinate of the system in angular form for (x_2,y_2) are

$$x_2 = x_1 + lsin(\theta_2)$$
$$= lsin(\theta_1) + lsin(\theta_2)$$
$$= l(sin(\theta_1) + sin(\theta_2))$$
$$y_2 = y_1 - lcos(\theta_2)$$
$$= -lcos(\theta_1) - lcos(\theta_2)$$
$$= -l(cos(\theta_1) + cos(\theta_2))$$

Since we are dealing with this system at a small angle, the trigonometry can be approximated using the Maclaurin series of expansion.

$$sin(\theta) \approx \theta$$
$$cos(\theta) \approx 1 - \frac{\theta^2}{2}$$
$$x_2 \approx l\theta_1 + l\theta_2$$
$$\dot{x_2} \approx l\dot{\theta}_1 + l\dot{\theta}_2 \tag{10}$$

$$y_2 \approx -l(1 - \frac{{\theta_1}^2}{2}) - l(1 - \frac{{\theta_2}^2}{2})$$
$$y_2 \approx \frac{l}{2}{\theta_1}^2 + \frac{l}{2}{\theta_2}^2 - 2l$$
$$\dot{y_2} \approx l\theta_1\dot{\theta}_1 + l\theta_2\dot{\theta}_2 \tag{11}$$

Under small angle oscillation, the displacement of the system in the y-axis is really small compared to the x coordinate, so the change in y coordinate displacement would be much smaller. In this case, the component of velocity for particle G_2 in the y coordinate would be treated as 0.

Substitute equation 10 back into equation 9, and the kinetic energy of particle G_2 will be

$$T = \frac{1}{2}m(l\dot{\theta}_1)^2 + \frac{1}{2}m(l^2{\dot{\theta}_1}^2 + l^2{\dot{\theta}_2}^2 + 2l\dot{\theta}_1 l\dot{\theta}_2)$$
$$T = \frac{1}{2}ml^2(2{\dot{\theta}_1}^2 + {\dot{\theta}_2}^2 + 2\dot{\theta}_1\dot{\theta}_2) \tag{12}$$

The potential energy of the system V can be found

$$V = mg(y_1 + y_2)$$
$$y_1 = -l(1 - \frac{{\theta_1}^2}{2})$$
$$y_2 \approx \frac{l}{2}{\theta_1}^2 + \frac{l}{2}{\theta_2}^2 - 2l$$
$$V \approx \frac{1}{2}mgl(2{\theta_1}^2 + {\theta_2}^2) \tag{13}$$

Substitute equation 12 and 13 back into equation 7, the Lagrangian for this system under small angle approximation will arrive.

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$$\boldsymbol{L} = \frac{1}{2} ml^2(2\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 + 2\dot{\theta}_1\dot{\theta}_2) - \frac{1}{2} mgl(2\theta_1{}^2 + \theta_2{}^2)$$

Then an Euler Lagrange equation will be used to determine the Equation of Motion of the system.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) \quad - \frac{\partial L}{\partial \theta_i} = 0 \quad \text{(i=1,2…)} \tag{14}$$

First start with angle θ_1

$$\frac{\partial L}{\partial \dot{\theta}_1} = \frac{1}{2} ml^2(4\dot{\theta}_1 + 2\dot{\theta}_2) = ml^2(2\dot{\theta}_1 + \dot{\theta}_2) \tag{15}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{1}{2} ml^2(2\dot{\theta}_1 + 2\dot{\theta}_2) = ml^2(\dot{\theta}_1 + \dot{\theta}_2) \tag{16}$$

$$\frac{\partial L}{\partial \theta_1} = -\frac{1}{2} mgl4\theta_1 = -2mgl\theta_1 \tag{17}$$

$$\frac{\partial L}{\partial \theta_2} = -mgl\theta_2 \tag{18}$$

For angle θ_1 the equation of motion is the derivative of equation 15 with respect to time minus the equation 17

$$\begin{gathered} ml^2(2\ddot{\theta}_1 + \ddot{\theta}_2) + 2mgl\theta_1 = 0 \\ 2\ddot{\theta}_1 + \ddot{\theta}_2 + 2\frac{g}{l}\theta_1 = 0 \end{gathered} \tag{19}$$

For angle θ_2 The equation of motion is the derivative of equation 16 with respect to time minus the equation 18.

$$\begin{gathered} ml^2(\ddot{\theta}_1 + \ddot{\theta}_2) + mgl\theta_2 = 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 + \frac{g}{l}\theta_2 = 0 \end{gathered} \tag{20}$$

The equations 19 and 20 are the equation of motion of the system, The small angle approximation has made them become linear. This means a single matrix equation can be written out and solved with the general solution of angle 1 and angle 2 with respect to the time t.

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1'' \\ \theta_2'' \end{bmatrix} + \frac{g}{l} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = 0 \tag{21}$$

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3. NORMAL MODES AND GENERAL MOTION OF THE SYSTEM IN SMALL ANGLE

Normal mode is a type of motion in which all parts of the system are oscillating with a constant angular frequency. These modes can then be used to process the motion of the double pendulum system, assuming the small angle approximation still holds.

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} cos(\omega t + \phi) \tag{22}$$

Equation 22 is the trial solution where C1 and C2 are constants. The ϕ here has no other meaning except showing a phase change.

If the trials solution is taken as a second derivative with respect to time

$$\begin{bmatrix} \theta_1'' \\ \theta_2'' \end{bmatrix} = -\omega^2 \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} cos(\omega t + \phi) \tag{23}$$

Substitute equations 23 and 22 back into 21

$$-\omega^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + \frac{g}{l} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = 0$$

A big matrix can be formed

$$\underbrace{\begin{bmatrix} 2g/l - 2\omega^2 & -\omega^2 \\ -\omega^2 & g/l - \omega^2 \end{bmatrix}}_{A} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = 0 \tag{24}$$

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \neq 0$$

In the case where the $\rightarrow$ Det A = 0

$$\text{Det(A)} = 2(\frac{g}{l} - \omega^2)^2 - \omega^4 = 0$$

$$\omega^4 - 4\omega^2 \frac{g}{l} + 2(\frac{g}{l})^2 = 0$$

$$(\omega^2 - \frac{2g}{l})^2 - 2(\frac{g}{l})^2 = 0$$

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$$\omega^2 = \frac{2g}{l} \pm \sqrt{2}\frac{g}{l} = \frac{g}{l}(2 \pm \sqrt{2})$$

$$\omega = \pm\sqrt{2 \pm \sqrt{2}}\sqrt{\frac{g}{l}} \tag{25}$$

The above is the equation of the frequencies of the normal modes for the double pendulum in small angles, or the eigenvalues.

The matrix for equation 24 can be converted into an algebraic format

$$-\omega^2 C_1 + (\frac{g}{l} - \omega^2)C_2 = 0$$

$$\frac{C_1}{C_2} = \frac{\omega^2}{(\frac{g}{l} - \omega^2)}$$

$$\frac{C_1}{C_2} = \frac{\omega^2 \frac{l}{g}}{(1 - \omega^2 \frac{l}{g})}$$

The value of angular frequency can be substituted with equation 25

$$\frac{C_1}{C_2} = \frac{2 \pm \sqrt{2}}{(-1 \mp \sqrt{2})} = \frac{-2 + 2 \pm \sqrt{2} \mp \sqrt{2}}{1 - 2} = \frac{\pm\sqrt{2}}{-1} = \mp\sqrt{2}$$

So the ratio of the two constant would be minus plus of root two depends on the sign of the angular frequency

After all the derive, the normal modes of the double pendulum can be written as two forms

$$\omega_+ = \sqrt{2 + \sqrt{2}}\sqrt{\frac{g}{l}}$$

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \propto \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$$

This eigenvector indicates a mode, which the movement of the two angles θ_1 and θ_2 are changing in the opposite direction toward the equilibrium or out of phase. This mode generally indicates the phase with higher frequency in the double pendulum.

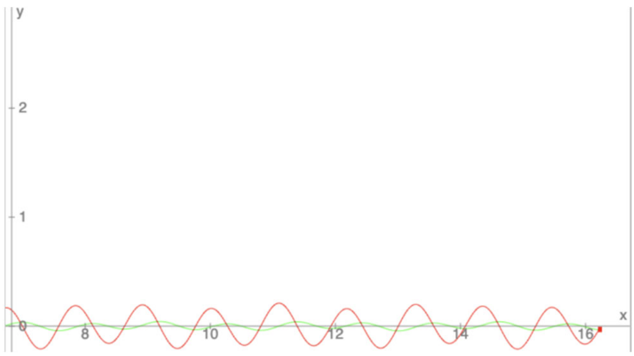

Figure 4. Positive frequency mode. Y axis indicates the size of the angle, x axis indicates the time. Green line is the graph of θ1 with respect to time and red line is the θ2 with respect to time.

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$$\omega_- = \sqrt{2-\sqrt{2}}\sqrt{\frac{g}{l}}$$

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \propto \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$$

This eigenvector indicates a mode, in which the movement of the two angles θ_1 and θ_2 are changing in the same direction toward the equilibrium or in phase. This mode generally indicates the phase with a lower frequency in the double pendulum.

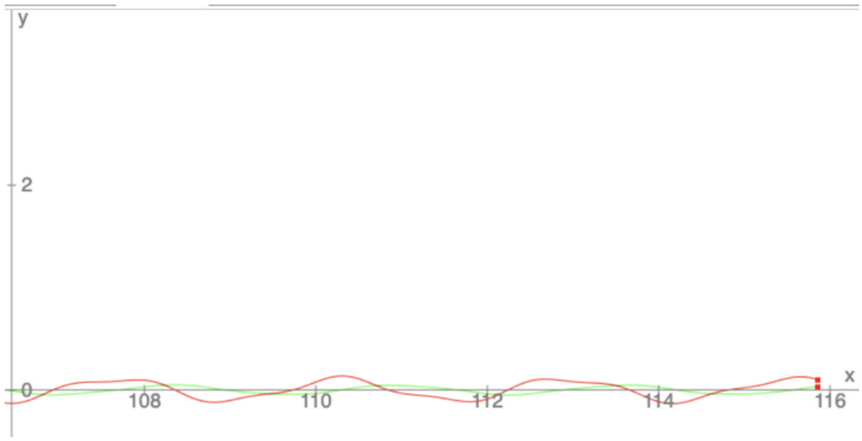

Figure 5. The negative frequency mode. Y axis indicates the size of the angle, x axis indicates the time. Green line is the graph of θ_1 with respect to time and red line is the θ_2 with respect to time.

The general solution can then be written as

$$\begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix} = A \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} \cos(\omega_+ t) + B \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \cos(\omega_- t) \quad (26)$$

The constant A and B depend on the initial conditions of the double pendulum, it tells us the composition of the system given its initial conditions, whether it is a positive or negative frequency mode.

Equation 26 can be written into two equations, each representing the solution of θ_1 *and* θ_2.

$$\theta_1(t) = A cos(\sqrt{2+\sqrt{2}}\sqrt{\frac{g}{l}}t) + B\ cos(\sqrt{2-\sqrt{2}}\sqrt{\frac{g}{l}}t) \quad (27)$$

$$\theta_2(t) = \sqrt{2}A cos(\sqrt{2+\sqrt{2}}\sqrt{\frac{g}{l}}t) - \sqrt{2}B\ cos(\sqrt{2-\sqrt{2}}\sqrt{\frac{g}{l}}t) \quad (28)$$

Equations 27 and 28 have shown that even under small angle approximation, the two angles still don’t behave as Simple harmonic motion, but a linear combination of these two equations can approach a simple harmonic motion.

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$$\sqrt{2}\theta_1 + \theta_2 = 2\sqrt{2}Acos(\sqrt{2+\sqrt{2}}\sqrt{\frac{g}{l}}t) \tag{29}$$

$$\sqrt{2}\theta_1 - \theta_2 = 2\sqrt{2}Bcos(\sqrt{2-\sqrt{2}}\sqrt{\frac{g}{l}}t) \tag{30}$$

This indicates that by giving a very specific initial condition, the entire system would be executed in a simple harmonic motion.

The value put into the equations is g (gravitational constant) = 9.8 m/s^2. l (the length of the rod) = 1m.

In equation 27 and 28, letting the constant B = 0, $\theta_1(t=0) = 15$. 15 degrees is the largest angle in the small angle approximation before the uncertainty reaches 1%.

$$\theta_2(t=0) = \sqrt{2}Acos(\sqrt{2+\sqrt{2}}\sqrt{\frac{g}{l}}t) = \sqrt{2}\theta_1 = 15\sqrt{2}$$

A=15

These value set change the equation into

$$\theta_1(t) = 15cos(\sqrt{2-\sqrt{2}}\sqrt{9.8}t))$$

$$\theta_2(t) = 15\sqrt{2}cos(\sqrt{2-\sqrt{2}}\sqrt{9.8}t)$$

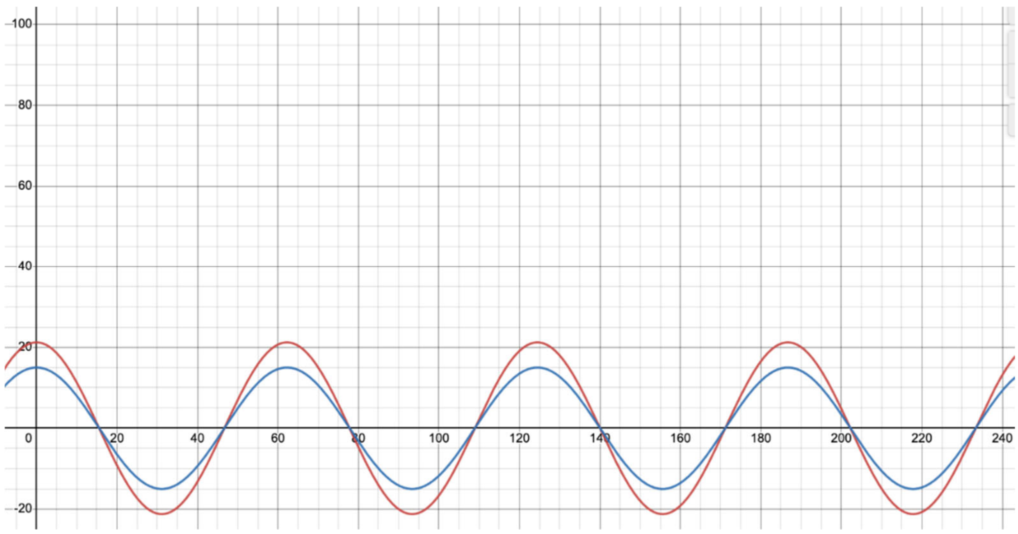

Figure 6. The blue line is the angle 1 with respect to time, red line is the angle 2 with respect to time under the initial value set into the function. Y axis indicates the size of the angle, x axis indicates the time.

Keeping those initial conditions, if constant A has been set up as 0.

$$\theta_1(t=0) = 15$$

$$\theta_2(t=0) = -\sqrt{2}Bcos(\sqrt{2+\sqrt{2}}\sqrt{\frac{g}{l}}t) = -\sqrt{2}\theta_1 = -15\sqrt{2}$$

B=15

These value set change the equation into

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$$\theta_1(t) \ = \ 15cos(\sqrt{2+\sqrt{2}}\sqrt{9.8}t))$$

$$\theta_2(t) \ = \ -15\sqrt{2}cos(\sqrt{2+\sqrt{2}}\sqrt{9.8}t)$$

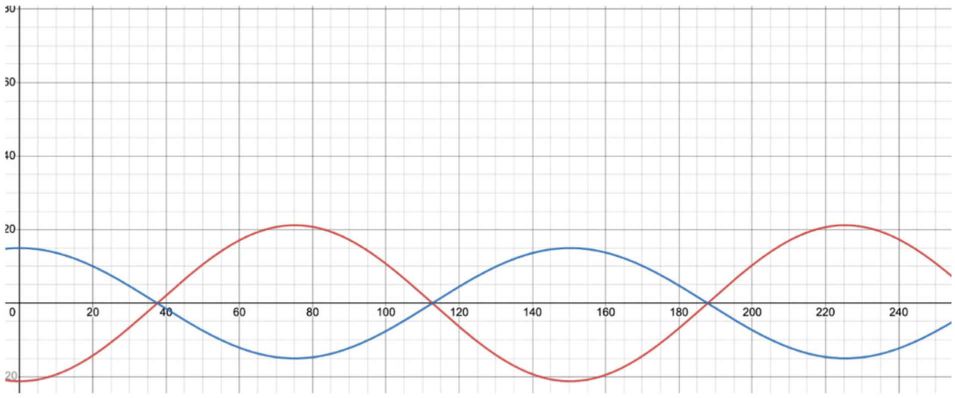

Figure 7. The blue line is the angle 1 with respect to time, red line is the angle 2 with respect to time under the initial value set into the function. Y axis indicates the size of the angle, x axis indicates the time.

Letting constant A or B as zero, this initialization has fully reduced one type of mode (as Figure 4 and 5 have shown), which transfers the whole system into a single mode, either in phase (figure 6) or out of phase (figure 7).

If A or B is not zero, the whole system would be transferred into the mixture of the two normal modes, which are less predictable and more chaotic shown in figure 8.

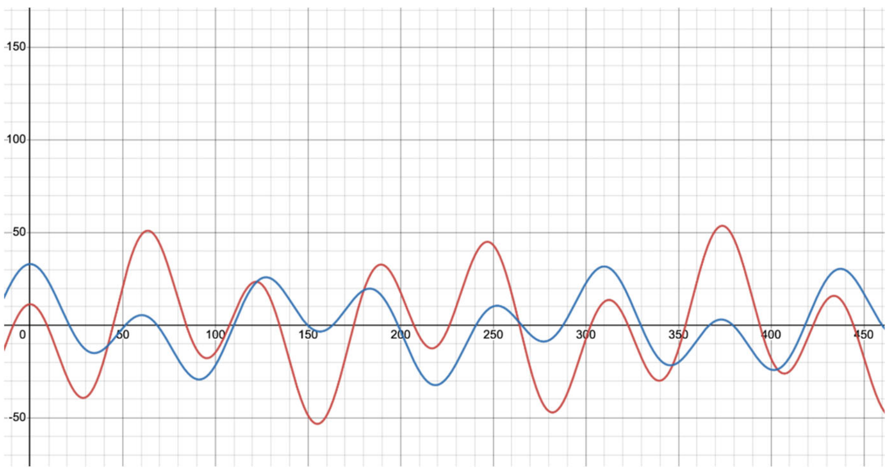

Figure 8. The blue line is the angle 1 with respect to time, red line is the angle 2 with respect to time under the initial value set into the function. Y axis indicates the size of the angle, x axis indicates the time.

The derivation of the formula is under small angle approximation, so part of the complicated constraints inside the system has been neglected. Ideally, the critical point of that transformation from ordered quasi-motion into chaotic motion would be unstable, but this can't be analysed using this small-angle model. This should be made using the formal, complete equation of motion of this double pendulum system, then used mechanical uncertainty to

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investigate this critical point. (since the complete equation of motion is nonlinear, so a direct numerical solution can't be processed out.)

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