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A Note on The Convergence of Generalized Integrals in The Absolute Value of A Function

Li Cao

School of Mathematics and Informatics, China West Normal University, China

Abstract

In the process of mathematical analysis learning, we know that if the function absolutely converges in the sense of generalized integral, the original function must converge. However, only if the absolute value of the function converges in the sense of generalized integral, such a conclusion does not necessarily hold. This article mainly addresses the problem of the convergence of such absolute functions on the convergence of the original function in the sense of generalized integrals, and can also expand the thinking about the effects between the convergence of the integral of other functions.

Keywords

The absolute value of the function converges; Absolute convergence; Infinite integrals; Defective points.

1. INTRODUCTION

Integrals are mainly the study of definite and indefinite integrals, and the generalized integral definition of functions is very important for calculus as a whole and for mathematical analysis. For definite integrals, we mainly solve in finite integral intervals and bounded integrable functions, while for our generalized integrals, we break through the limitations of these conditions and consider infinite intervals or unbounded functions, so its convergence will be more complex than definite integrals.

It is noted that the influence of absolute convergence in the sense of generalized integrals on the convergence of the original function in the sense of generalized integrals is not discussed too much, but in fact, after the exploration of this paper, it is found that this proposition is not necessarily true, and at the same time, more integral convergence in the form of generalized integrals can be explored in relation to the convergence of the original function under the definition of generalized integrals.

2. PREREQUISITES

The following definition is taken from [1] page 248 of the first volume of the textbook Mathematical Analysis of ECNU Fifth Edition.

Definition 1 Suppose functions f is defined on the infinite interval $[a,+\infty)$, and can be integrable on any finite interval $[a,u]$,if there is a limit

$$\lim_{u\to+\infty}\int_a^u f(x)dx = J$$

this limit J is called the infinite anomalous integral of the function f over the interval（or infinite integrals）,note as

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$$J=\int_a^{+\infty} f(x)dx.$$

And called $\int_a^{+\infty} f(x)dx$ is convergent. If the limit does not exist, for convenience, also known as $\int_a^{+\infty} f(x)dx$ is divergence.

Similar to define the infinite integral of function f over an infinite interval $(-\infty,b]$.

Definition 2 Suppose functions f is defined on the infinite interval $[a,+\infty)$, and can be integrable on any finite interval $[a,u]$, let $g(x)=|f(x)|$, if there is a limit

$$\lim_{u\to\infty}\int_a^u g(x)dx=J$$

note as

$$J=\int_a^{+\infty} g(x)dx$$

and called $\int_a^{+\infty} g(x)dx$ is absolute convergent. If the limit does not exist, for convenience, also known as $\int_a^{+\infty} g(x)dx$ is divergence.

Definition 3 Suppose functions f is defined on the infinite interval $(a,b]$, unbounded on either right neighborhood of point a, however, it is bounded and integrable in any closed interval $[u,b]\subset(a,b]$, if there is a limit

$$\lim_{u\to a^+}\int_u^b f(x)dx=J$$

this limit is called the anomalous integral of the unbounded function f on $(a,b]$,note as

$$J=\int_a^b f(x)dx$$

and called anomaly integrals $\int_a^b f(x)dx$ is convergent. If the limit does not exist, called anomaly integrals $\int_a^b f(x)dx$ is divergence.

The integrand f is defined unbounded in point a vicinity, at this time, call point a is the flaw of integrand f, and anomaly integrals of unbounded functions $\int_a^b f(x)dx$ also known as defective integral. Similarly, when the flaw points is b:

$$\int_a^b f(x)dx=\lim_{u\to b^-}\int_a^u f(x)dx$$

3. PROPERTIES

For most cases, convergence between the various forms of functions is inconsistent in the sense of generalized integrals.

Nature 1

(1) When $\int_a^{+\infty} f(x)dx$ is convergent, $\int_a^{+\infty} f^2(x)dx$ is not necessarily convergent;

(2) When $\int_a^{+\infty} f^2(x)dx$ is convergent, $\int_a^{+\infty} f(x)dx$ is not necessarily convergent.

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Example 1 Let $f(x)=\frac{\sin x}{\sqrt{x}}$, from Dirichlet criterions[2], we have $\int_1^{+\infty}\frac{\sin x}{\sqrt{x}}dx$ is convergent, but $\int_1^{+\infty}\frac{\sin^2 x}{x}dx$ is divergence.

Example 2 Let $f(x)=\frac{1}{x^{\frac{2}{3}}}$, we have $\int_1^{+\infty}f^2(x)dx=\int_1^{+\infty}\frac{1}{x^{\frac{4}{3}}}dx$ is convergent, but $\int_1^{+\infty}\frac{1}{x^{\frac{2}{3}}}dx$ is divergence.

Nature 2

(1) When $\int_a^{+\infty}|f(x)|dx$ is convergent, $\int_a^{+\infty}f^2(x)dx$ is not necessarily convergent;

(2) When $\int_a^{+\infty}f^2(x)dx$ is convergent, $\int_a^{+\infty}|f(x)|dx$ is not necessarily convergent.

Example 3 Let $f(x)=\begin{cases}0 & x\in\left[n-1,n-\frac{1}{4^n}\right]\\(-2)^n & x\in\left(n-\frac{1}{4^n},n\right)\end{cases}$, we have $\int_0^{+\infty}|f(x)|dx=\sum_{n=1}^{+\infty}\frac{2^n}{4^n}=1$ is convergent, but $\int_0^{+\infty}f^2(x)dx=\sum_{n=1}^{+\infty}\frac{4^n}{4^n}=+\infty$ is divergence.

Example 4 Let $f(x)=\frac{1}{x^{\frac{2}{3}}}$, we have $\int_1^{+\infty}f^2(x)dx=\int_1^{+\infty}\frac{1}{x^{\frac{4}{3}}}dx$ is convergent, but $\int_1^{+\infty}\frac{1}{x^{\frac{2}{3}}}dx$ is divergence.

Nature 3 For defective points we have $\int_a^b f(x)dx$ is convergent, but $\int_a^b f^2(x)dx$ is not necessarily convergent.

Example 5 Let $f(x)=\frac{1}{\sqrt{x}}$, we have $\int_0^1 f(x)dx=\int_0^1\frac{1}{\sqrt{x}}dx$ is convergent, but $\int_0^1 f^2(x)dx=\int_0^1\frac{1}{x}dx$ is divergence.

4. MAIN CONCLUSIONS

Nature 4 If the function f is integrable on any finite interval $[a,u]$, and $\int_a^{+\infty}|f(x)|dx$ is convergent, then $\int_a^{+\infty}f(x)dx$ must also is convergent and has

$$\left|\int_a^{+\infty}f(x)dx\right|\le\int_a^{+\infty}|f(x)|dx$$

To prove completeness, the following proof is taken from page 252 of the first volume of the textbook Mathematical Analysis of East China Normal University.

Proof Due to $\int_a^{+\infty}|f(x)|dx$ is convergent, according to the Cauchy convergence criterion[3](necessity), for arbitrary $\varepsilon>0$,exist $G\ge a$,when $u_2>u_1>G$,there is always

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$$\left|\int_{u_1}^{u_2}|f(x)|dx\right|=\int_{u_1}^{u_2}|f(x)|dx<\varepsilon$$

And using the absolute value inequality of definite integrals, so there is

$$\left|\int_{u_1}^{u_2}f(x)dx\right|\le\int_{u_1}^{u_2}|f(x)|dx<\varepsilon$$

then by Cauchy criterion(sufficiency), $\int_a^{+\infty}f(x)dx$ convergence certified.

Since $\left|\int_a^u f(x)dx\right|\le\int_a^u|f(x)|dx \quad (u>a)$, let $u\to+\infty$ take the limit, the above inequality is immediately obtained.

Naturally, we note the conditions in the above nature "If the function f is integrable on any finite interval $[a,u]$", could this be deleted.

That is, when $\int_a^{+\infty}|f(x)|dx$ is convergent, $\int_a^{+\infty}f(x)dx$ is also convergent, the results are not necessarily convergent, the following are examples of this conclusion.

Nature 5

(1) When $\int_a^{+\infty}|f(x)|dx$ is convergent, $\int_a^{+\infty}f(x)dx$ is not necessarily convergent;

(2) When $\int_a^{+\infty}f(x)dx$ is convergent, $\int_a^{+\infty}|f(x)|dx$ is not necessarily convergent.

Example 6 Suppose function

$$f(x)=\begin{cases}1 & x\in[1,2]\cap Q\\ -1 & x\in[1,2]\setminus Q\\ \dfrac{1}{x^2} & x\in(2,+\infty)\end{cases},$$

it can be obtained

$$|f(x)|=\begin{cases}1 & x\in[1,2]\\ \dfrac{1}{x^2} & x\in(2,+\infty)\end{cases}$$

It proves that $\int_a^{+\infty}|f(x)|dx$ is convergent, but $\int_a^{+\infty}f(x)dx$ is not convergent.

Proof Let $g(x)=|f(x)|$, we have

$$\int_1^{+\infty}g(x)dx=\int_1^{+\infty}|f(x)|dx=x\Big|_1^2+\left(-\frac{1}{x}\right)\Big|_2^{+\infty}=1+0=1$$

We obtain

$\int_1^{+\infty}g(x)dx$ is convergent.

But apparently the original function is not convergent, so if there is no "If the function f is integrable on any finite interval $[a,u]$", this nature does not necessarily hold, other words: When $\int_a^{+\infty}|f(x)|dx$ is convergent, $\int_a^{+\infty}f(x)dx$ is not necessarily convergent.

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Consider that defective integrals and infinite integrals are interchangeable and have parallel theories and results [4], so we reasonably speculate, the same is true for defective points.

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