Generalized Fuzzy DEA/AR Model Based on Monte Carlo Simulation and Its Application

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Abstract

The traditional DEA regards the DMU group as a black box and only focuses on the relative efficiency of the DMU group. Stackelberg (leader-follower) game theory is used to give priority to and successively decompose the efficiency of the DMU into the efficiency value of its own DMU. In this paper, a new two-stage Stackelberg fuzzy DEA model is proposed to calculate the efficiency values of DMUs and sub-DMUs. The model has three characteristics: first, it is linear, which makes it possible to quickly obtain the global optimal solution; secondly, the model is independent of the α cut set variables, which greatly reduces the calculation amount of the model and improves the accuracy of efficiency calculation; thirdly, the paper uses model Carlo simulation to rank the decision-making units. Finally, the efficiency of public cultural services in 13 cities of Beijing-Tianjin-Hebei urban agglomeration is calculated to verify the effectiveness of the model.

Keywords

Generalized fuzzy DEA; Monte Carlo simulation; Beijing-Tianjin-Hebei urban agglomeration; Public cultural services.

1. INTRODUCTION

DEA is an effective nonparametric method to measure the relative efficiency of homogeneous DMUs. Since 1978, the application of DEA model has been continuously diversified. The four most widely used models are DEA-CCR [1] (return to scale invariant model), DEA-BCC [2] (return to scale variable model), DEA-FG [3] (return to scale non-increasing model) and DEA-ST [4] (return to scale non-decreasing model). These four models will be limited by whether the returns to scale is variable in use. Based on this, Ghasemi [5] proposed a generalized DEA model, these four DEA models are included to make them widely applicable. These four traditional DEA models and Ghasemi's generalized DEA model have "one premise and two implicit assumptions", which limits the promotion and use of DEA model.

First of all, "one premise" means that the input-output factors of the decision-making unit must be accurately measured by figures. However, some input-output factors are difficult to measure accurately and can only be analyzed vaguely, such as cultural impact, staff effort and staff quality. Some scholars have proposed to construct fuzzy DEA (FDEA) model to deal with fuzzy input-output factors by using fuzzy sorting method, de-fuzzy method and α cut set method. Among them, the α cut set method is the most widely used DEA model with fuzzy input-output factors, but its calculation procedure of α cut set-DEA model is complex, which limits its practical application.

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Secondly, the first "implicit hypothesis" means that the efficiency of the decision-making unit calculated by the DEA model is a weighted efficiency. When calculating, it will automatically select the optimal proportion of input factors for the input factors of each decision-making unit to obtain the best output efficiency. However, in actual production, due to the limitations of the effectiveness of production resources such as capital and human resources or production conditions, the input factors of decision-making units are often not able to produce at the optimal proportion of input, but are limited to a certain amount. Based on this, the concept of assurance domain (AR) was introduced to limit the proportion of input factors to a reasonable range. Liu [7] introduced the concept of AR into the DEA model and used it to measure the production efficiency of manufacturing enterprises with constant return to scale (CCR). However, their research ignored the constraints of production conditions of DMUs. Jain [8] took the constraints of the decision-making unit into account, and proved the upper and lower limits of efficiency at different levels. However, the above studies regard it as a priori hypothesis, ignoring the verification of the guarantee domain constraints.

Finally, the "second implicit hypothesis" means that the efficiency of the decision-making units of the DEA model is a relative efficiency. According to the results, the decision-making units can only be divided into effective decision-making units and invalid decision-making units. It is difficult to rank them according to the efficiency results of the decision-making units.

Based on this, in view of the limitations of the use of DEA/AR model and the problem that the ranking method of DEA model is divorced from reality, this paper will build a generalized fuzzy DEA/AR model based on Monte Carlo simulation on the basis of GDEA/AR proposed by Ghasemi [5] tosolve the above problems.

2. INTRODUCTION OF GENERALIZED DEA MODEL

Assuming that there are many decision-making units DMU_j ($j=1,2,\ldots,n$), X_{ij} and Y_{ij} are the input-output factors of the j decision-making unit,according to YU's generalized DEA model, its linear programming model is shown in formula (1):

$$E_{j_o} = Max(\sum_{r=1}^{s} u_r Y_{rj_o}) - \delta_1 u_o$$

$$st.\quad \begin{cases} \sum_{i=1}^{m} v_i X_{ij_o} = 1, i = 1,2,\ldots,m, \\ \sum_{r=1}^{s} u_r Y_{rj} - \sum_{i=1}^{m} v_i X_{ij} - \delta_1 u_o \le 0, \quad j = 1,2,\ldots,n, \\ \delta_1 \delta_2 (-1)^{\delta_3} u_o \ge 0, \\ v_i \ge \varepsilon, u_r \ge \varepsilon, r = 1,\ldots,s, \end{cases} \tag{1}$$

Among them, $\varepsilon > 0$ is an infinitesimal non-Archimedean number, δ_i is a binary parameter with a value of 0 or 1, v_i is the weight of input element X_{ij} , u_r is the weight of output factor y_{ij} ,If $\delta_1 = 0$, GDEA model becomes CCR-DEA model. If $\delta_1 = 1$ and $\delta_2 = 0$, GDEA model becomes BCC-DEA model. If $\delta_1 = 1$, $\delta_2 = 1$ and $\delta_3 = 1$, GDEA becomes the ST-DEA model. Moreover, v_i and u_r are the unknowns in the GDEA model, and each decision-making unit will choose the most favorable weight combination for input and output factors.

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3. CONSTRUCTION AND ALGORITHM DESIGN OF GENERALIZED FUZZY DEA/AR MODEL

3.1. Construction of generalized DEA/AR model

However, due to the constraints of production conditions, in many cases, the weight combination value of input-output factors must be limited. The concept of guarantee region (AR) is introduced to limit the weight of DEA model elements. This paper will use the AR concept proposed in literature (23), add the AR concept to the GDEA model, and obtain the generalized DEA/AR model as shown in (2):

$$E_{j_o} = Max(\sum_{r=1}^{s} u_r Y_{rj_o}) - \delta_1 u_o$$

$$st.\begin{cases} \sum_{i=1}^{m} v_i X_{ij_o} = 1, i = 1,2,\ldots,m, \\ \sum_{r=1}^{s} u_r Y_{rj} - \sum_{i=1}^{m} v_i X_{ij} - \delta_1 u_o \le 0, \quad j = 1,2,\ldots,n, \\ C_{pq}^L v_q \le v_p \le C_{pq}^U v_q \\ D_{pq}^L u_q \le u_p \le D_{pq}^U u_q \\ v_i \ge \varepsilon, u_r \ge \varepsilon, r = 1,\ldots,s, \\ \delta_1 \delta_2 (-1)^{\delta_3} u_o \ge 0, \end{cases} \tag{2}$$

The meaning of each parameter is the same as above.

3.2. Construction of generalized fuzzy DEA/AR model

This article needs to be defined as follows.

Definition 1:The α -cut set $\bar{A}_\alpha$ of fuzzy set $\bar{A}$ is defined as: $\bar{A}_\alpha = \{x_i : \mu_A(x_i) \ge \alpha, x_i \in X\}$, $\alpha \in [0,1]$ .On the basis of the above definition, $\tilde{X}_{ij}$ and $\tilde{Y}_{ij}$ are used to represent the fuzzy numbers of input and output elements, and their membership functions are $\mu_{\tilde{X}_{ij}}$ and $\mu_{\tilde{Y}_{ij}}$. The linear programming model of GFDEA based on guarantee region (AR) is shown in formula (3):

$$\tilde{E}_{j_o} = Max(\sum_{r=1}^{s} u_r \tilde{Y}_{rj_o}) - \delta_1 u_o$$

$$st.\begin{cases} \sum_{i=1}^{m} v_i \tilde{X}_{ij_o} = 1, i = 1,2,\ldots,m, \\ \sum_{r=1}^{s} u_r \tilde{Y}_{rj} - \sum_{i=1}^{m} v_i \tilde{X}_{ij} - \delta_1 u_o \le 0 \\ C_{pq}^L v_q \le v_p \le C_{pq}^U v_q, 1 \le p < q = 2,\ldots m \\ D_{pq}^L u_q \le u_p \le D_{pq}^U u_q, 1 \le p < q = 2,\ldots m \\ v_i \ge \varepsilon, u_r \ge \varepsilon, r = 1,\ldots,s, \\ \delta_1 \delta_2 (-1)^{\delta_3} u_o \ge 0, j = 1,2,\ldots,n \end{cases} \tag{3}$$

Obviously, model (3) can produce fuzzy efficiency values, and its lowest $(E_{j_o})_\alpha^L$ and the highest level solution $(E_{j_o})_\alpha^U$ can be obtained by equations (4) and (5):

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$$(E_{j_o})_\alpha^U = \max_{\substack{x_{ij}\in(\tilde{X}_{ij})_\alpha \\ y_{rj}\in(\tilde{Y}_{rj})_\alpha}} \begin{cases} Max(\sum_{r=1}^{s} u_r \tilde{Y}_{rj_o}) - \delta_1 u_o \\ st. \begin{cases} \sum_{i=1}^{m} v_i \tilde{X}_{ij_o} = 1, i = 1,2,\ldots,m, \\ \sum_{r=1}^{s} u_r \tilde{Y}_{rj} - \sum_{i=1}^{m} v_i \tilde{X}_{ij} - \delta_1 u_o \le 0, \\ C_{pq}^L v_q \le v_p \le C_{pq}^U v_q \\ D_{pq}^L u_q \le u_p \le D_{pq}^U u_q \\ v_i \ge \varepsilon, u_r \ge \varepsilon, r = 1,\ldots,s, \\ \delta_1 \delta_2 (-1)^{\delta_3} u_o \ge 0, \end{cases} \end{cases} \tag{4}$$

$$(E_{j_o})_\alpha^L = \min_{\substack{x_{ij}\in(\tilde{x}_{ij})_\alpha \\ y_{rj}\in(\tilde{y}_{rj})_\alpha}} \begin{cases} Max(\sum_{r=1}^{s} u_r \tilde{Y}_{rj_o}) - \delta_1 u_o \\ st. \begin{cases} \sum_{i=1}^{m} v_i \tilde{X}_{ij_o} = 1, i = 1,2,\ldots,m, \\ \sum_{r=1}^{s} u_r \tilde{Y}_{rj} - \sum_{i=1}^{m} v_i \tilde{X}_{ij} - \delta_1 u_o \le 0, \\ C_{pq}^L v_q \le v_p \le C_{pq}^U v_q \\ D_{pq}^L u_q \le u_p \le D_{pq}^U u_q \\ v_i \ge \varepsilon, u_r \ge \varepsilon, r = 1,\ldots,s, \\ \delta_1 \delta_2 (-1)^{\delta_3} u_o \ge 0, \end{cases} \end{cases} \tag{5}$$

3.3. Algorithm design of generalized fuzzy DEA/AR model

(1) Convert nonlinear model into linear programming model

Equations (4) and (5) are two-level nonlinear programming models, which cannot directly find the optimal solution. They need to be converted into linear programming models. First, theorem 1 is proposed.

Theorem 1: The optimal solution of model (4) is the same as that of equation (6).

$$(E_{j_o})_\alpha^U = \max \sum_{r=1}^{s} u_r (Y_{rj_o})_\alpha^U - \delta_1 u_o$$

$$st. \begin{cases} \sum_{i=1}^{m} v_i (X_{ij_o})_\alpha^L = 1, i = 1,2,\ldots,m, \\ \sum_{r=1}^{s} u_r (Y_{rj_o})_\alpha^U - \sum_{i=1}^{m} v_i (X_{ij_o})_\alpha^L - \delta_1 u_o \le 0 \\ \sum_{r=1}^{s} u_r (Y_{rj})_\alpha^L - \sum_{i=1}^{m} v_i (X_{ij})_\alpha^U - \delta_1 u_o \le 0 \\ C_{pq}^L v_q \le v_p \le C_{pq}^U v_q \\ D_{pq}^L u_q \le u_p \le D_{pq}^U u_q \\ v_i \ge \varepsilon, u_r \ge \varepsilon, r = 1,\ldots,s, \\ \delta_1 \delta_2 (-1)^{\delta_3} u_o \ge 0, j = 1,2,\ldots,n, j \ne j_o \end{cases} \tag{6}$$

Proof: Assume that the decision-making unit j_o is the unit to be evaluated, and $\forall x_{ij} \in \left[(X_{ij})_\alpha^L, (X_{ij})_\alpha^U\right], y_{ij} \in \left[(Y_{ij})_\alpha^L, (Y_{ij})_\alpha^U\right]$, construct the following model: $E_{j_o}^\alpha = \max \sum_{r=1}^{s} u y_{rj_o} - \delta_1 u_o$

$$st.\begin{cases}\sum_{i=1}^{m} v_i x_{ij_o} = 1, i = 1,2,\ldots,m, \\ \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - \delta_1 u_o \le 0, \\ C_{pq}^{L} v_q \le v_p \le C_{pq}^{U} v_q \\ D_{pq}^{L} u_q \le u_p \le D_{pq}^{U} u_q \\ v_i \ge \varepsilon, u_r \ge \varepsilon, r = 1,\ldots,s, \\ \delta_1 \delta_2 (-1)^{\delta_3} u_o \ge 0, j = 1,2,\ldots,n \end{cases} \quad (7)$$

If the optimal solution of equation (7) is $(u_o^*, U^*, V^*) = (u_o^*, u_1^* \ldots u_s^*, v_1^* \ldots v_m^*)$, let $u_o^{'} = u_o^*$, $u_r^{'} = u_r^* (\sum_{r=1}^{s} u_r^* y_{rj_o} / \sum_{r=1}^{s} u_r^* (Y_{rj_o})_\alpha^U)$, $v_i^{'} = v_i^* (\sum_{i=1}^{m} v_i^* x_{ij_o} / \sum_{i=1}^{m} v_i^* (X_{ij_o})_\alpha^L)$, then it is proved that $(u_o^{'}, U^{'}, V^{'})$ is also the feasible solution of equation (6).

First, take $(u_o^{'}, U^{'}, V^{'})$ into equation (7) and get:

$$\sum_{i=1}^{s} v_i^{'} (X_{ij_o})_\alpha^L = \frac{\sum_{i=1}^{m} v_i^* x_{ij_o}}{\sum_{i=1}^{m} v_i^* (X_{ij_o})_\alpha^L} \sum_{i=1}^{s} v_i^* (X_{ij_o})_\alpha^L = \sum_{i=1}^{m} v_i^* x_{ij_o} = 1,$$

$$\sum_{r=1}^{s} u_r^{'} (Y_{rjo})_\alpha^U - \sum_{i=1}^{m} v_i^{'} (X_{ijo})_\alpha^L - \delta_1 u_o^{'} = \frac{\sum_{r=1}^{s} u_r^* y_{rj_o}}{\sum_{r=1}^{s} u_r^* (Y_{rj_o})_\alpha^U} \sum_{r=1}^{s} u_r^* (Y_{rjo})_\alpha^U - \frac{\sum_{i=1}^{m} v_i^* x_{ij_o}}{\sum_{i=1}^{m} v_i^* (X_{ij_o})_\alpha^L} \sum_{i=1}^{m} v_i^* (X_{ijo})_\alpha^L - \delta_1 u_o^* = \sum_{r=1}^{s} u_r^* y_{rj_o} - \sum_{i=1}^{m} v_i^* x_{ij_o} - \delta_1 u_o^* \le 0$$

.Because of $y_{rjo} \le (Y_{rjo})_\alpha^U$ and $x_{ij_o} \ge (X_{ij_o})_\alpha^L$, therefore:

$u_r^{'} = u_r^* \frac{\sum_{r=1}^{s} u_r^* y_{rj_o}}{\sum_{r=1}^{s} u_r^* (Y_{rj_o})_\alpha^U} \le u_r^*$, $v_i^{'} = v_i^* \frac{\sum_{i=1}^{m} v_i^* x_{ij_o}}{\sum_{i=1}^{m} v_i^* (X_{ij_o})_\alpha^L} \ge v_i^*$,so, $\sum_{r=1}^{s} u_r^{'} (Y_{rj})_\alpha^L \le \sum_{r=1}^{s} u_r^* y_{rj} \le \sum_{i=1}^{m} v_i^* x_{ij_o} + \delta_1 u_o^* \le \sum_{i=1}^{m} v_i^{'} (X_{ij})_\alpha^U + \delta_1 u_o^{'} \le 0$, $\sum_{r=1}^{s} u_r^{'} (Y_{rj})_\alpha^L - \sum_{i=1}^{m} v_i^{'} (X_{ij})_\alpha^U - \delta_1 u_o^{'} \le 0$.

Secondly, for the assurance area (AR), it is obvious that:

$$\frac{v_p^{'}}{v_q^{'}} = v_p^* \Big(\frac{\sum_{i=1}^{m} v_i^* x_{ij_o}}{\sum_{i=1}^{m} v_i^* (X_{ij_o})_\alpha^L}\Big) / v_q^* \Big(\frac{\sum_{i=1}^{m} v_i^* x_{ij_o}}{\sum_{i=1}^{m} v_i^* (X_{ij_o})_\alpha^L}\Big) = \frac{v_p^*}{v_q^*}, \quad \frac{u_p^{'}}{u_q^{'}} = u_p^* \Big(\frac{\sum_{r=1}^{s} u_r^* y_{rj_o}}{\sum_{r=1}^{s} u_r^* (Y_{rj_o})_\alpha^U}\Big) / u_q^* \Big(\frac{\sum_{r=1}^{s} u_r^* y_{rj_o}}{\sum_{r=1}^{s} u_r^* (Y_{rj_o})_\alpha^U}\Big) = \frac{u_p^*}{u_q^*}$$

Therefore, $C_{pq}^L \le \frac{v_p^{'}}{v_q^{'}} \le C_{pq}^U$, $D_{pq}^L \le \frac{u_p^{'}}{u_q^{'}} \le D_{pq}^U$.

$\delta_1 \delta_2 (-1)^{\delta_3} u_o^{'} = \delta_1 \delta_2 (-1)^{\delta_3} u_o^* \ge 0$, because ε is a non-Archimedean infinite fraction, so $u_r^{'} \ge \varepsilon$, $v_i^{'} \ge \varepsilon$.

It can be seen that $(u_o^{'}, U^{'}, V^{'})$ is a feasible solution of equation (6), and its corresponding objective function value is: $\sum_{r=1}^{s} u_r^{'} (Y_{rj0})_\alpha^U - \delta_1 u_0^{'} = \sum_{r=1}^{s} u_r^* \frac{\sum_{r=1}^{s} u_r^* y_{rj_0}}{\sum_{r=1}^{s} u_r^* (Y_{rj_o})_\alpha^U} (Y_{rj_o})_\alpha^U - \delta_1 u_0^* = \sum_{r=1}^{s} u_r^* y_{rj_0} - \delta_1 u_0^* = E_{j_0}^\alpha$

Therefore, there is $E_{j_0}^\alpha \le (E_{j_0})_\alpha^U$, which means that if the input and output factors come from $(\tilde{X}_{ij})_\alpha$ and $(\tilde{Y}_{ij})_\alpha$, then the model (6) can obtain the maximum efficiency value, while the equation (4) is the maximum value under the α -cut solution level. It can be seen that the model (4) and the model(6) will obtain the same optimal value at the α level, and the solutions ofthe two models are the same, as proved by theorem 1.

Theorem 2: The optimal value of model (5) is the same as that of equation (8).

$$(E_{j_o})_\alpha^L = \max\sum_{r=1}^{s} u_r (Y_{rj_o})_\alpha^L - \delta_1 u_o \quad st. \quad \begin{cases} \sum_{i=1}^{m} v_i (x_{ij_0})_\alpha^U = 1, i = 1,2,\ldots,m, \\ \sum_{r=1}^{s} u_r (Y_{rj_o})_\alpha^L - \sum_{i=1}^{m} v_i (X_{ij_0})_\alpha^U - \delta_1 u_o \le 0, \\ \sum_{r=1}^{s} u_r (Y_{rj})_\alpha^U - \sum_{i=1}^{m} v_i (X_{ij})_\alpha^L - \delta_1 u_o \le 0, \\ C_{pq}^L v_q \le v_p \le C_{pq}^U v_q \\ D_{pq}^L u_q \le u_p \le D_{pq}^U u_q \\ v_i \ge \varepsilon, u_r \ge \varepsilon, r = 1,\ldots,s, \\ \delta_1\delta_2(-1)^{\delta_3} u_o \ge 0, j \ne j_o \end{cases} \tag{8}$$

Proof: Suppose is the optimal value of model $(\tilde{u}_0,\tilde{U},\tilde{V}) = (\tilde{u}_0,\tilde{u}_1,\ldots,\tilde{u}_s,\tilde{v}_1,\ldots,\tilde{v}_m)$ (8), and its corresponding objective function value is $(E_{j_o})_\alpha^L$.Let $\hat{u}_0 = \tilde{u}_0$, $\hat{u}_r = \tilde{u}_r \frac{\sum_{r=1}^{s}\tilde{u}_r (Y_{rj_o})_\alpha^L}{\sum_{r=1}^{s}\tilde{u}_r y_{rj_o}}$, $\hat{v}_i = \tilde{v}_i \frac{\sum_{i=1}^{m}\tilde{v}_i (X_{ij_o})_\alpha^U}{\sum_{i=1}^{m}\tilde{v}_i x_{ij_o}}$, the next part of this paper will prove that $(\tilde{u}_0,\tilde{U},\tilde{V})$ is also a feasible solution of model (7).

First, take $(\tilde{u}_0,\tilde{U},\tilde{V})$ into equation (8) and get, $\sum_{i=1}^{m}\hat{v}_i x_{ij_o} = \frac{\sum_{i=1}^{m}\tilde{v}_i (X_{ij_o})_\alpha^U}{\sum_{i=1}^{m}\tilde{v}_i x_{ij_o}} \sum_{i=1}^{m}\tilde{v}_i x_{ij_o} = \sum_{i=1}^{m}\tilde{v}_i (X_{ij_o})_\alpha^U = 1$, $\sum_{r=1}^{s}\hat{u}_r y_{rj_o} - \sum_{i=1}^{m}\hat{v}_i x_{ij_o} - \delta_1\hat{u}_0 = \sum_{r=1}^{s}\tilde{u}_r (Y_{rj_o})_\alpha^L - \sum_{i=1}^{m}\tilde{v}_i (X_{ij_o})_\alpha^U - \delta_1\tilde{u}_0 \le 0$,Because of $y_{rj_o} \ge (Y_{rj_o})_\alpha^L$ and $x_{ij_o} \le (X_{ij_o})_\alpha^U$, there are: $\hat{u}_r = \tilde{u}_r \frac{\sum_{r=1}^{s}\tilde{u}_r (Y_{rj_o})_\alpha^L}{\sum_{r=1}^{s}\tilde{u}_r y_{rj_0}} \le \tilde{u}_r$, $\hat{v}_i = \tilde{v}_i \frac{\sum_{i=1}^{m}\tilde{v}_i (X_{ij_o})_\alpha^U}{\sum_{i=1}^{m}\tilde{v}_i x_{ij_0}} \ge \tilde{v}_i$,So, $\sum_{r=1}^{s}\hat{u}_r y_{rj} \le \sum_{r=1}^{s}\tilde{u}_r (Y_{rj})_\alpha^U \le \sum_{i=1}^{m}\tilde{v}_i (X_{ij})_\alpha^L + \delta_1\tilde{u}_0 \le \sum_{i=1}^{m}\hat{v}_i x_{ij} + \delta_1\hat{u}_0$,Therefore, $\sum_{r=1}^{s}\hat{u}_r y_{rj} - \sum_{i=1}^{m}\tilde{v}_i x_{ij} - \delta_1\hat{u}_0 \le 0$.

Secondly, for the assurance area (AR), it is obvious that: $\frac{\hat{v}_p}{\hat{v}_q} = \tilde{v}_p \left(\frac{\sum_{i=1}^{m}\tilde{v}_i (X_{ij_0})_\alpha^U}{\sum_{i=1}^{m}\tilde{v}_i x_{ij_o}}\right) / \tilde{v}_q \left(\frac{\sum_{i=1}^{m}\tilde{v}_i (X_{ij_0})_\alpha^U}{\sum_{i=1}^{m}\tilde{v}_i x_{ij_o}}\right) = \frac{\tilde{v}_p}{\tilde{v}_q}$, $\frac{\hat{u}_p}{\hat{u}_q} = \tilde{u}_p \left(\frac{\sum_{r=1}^{s}\tilde{u}_r (Y_{rj_o})_\alpha^L}{\sum_{r=1}^{s}\tilde{u}_r y_{rj_0}}\right) / \tilde{u}_q \left(\frac{\sum_{r=1}^{s}\tilde{u}_r (Y_{rj_o})_\alpha^L}{\sum_{r=1}^{s}\tilde{u}_r y_{rj_0}}\right) = \frac{\tilde{u}_p}{\tilde{u}_q}$.

So, $C_{pq}^L \le \frac{\hat{v}_p}{\hat{v}_q} \le C_{pq}^U$, $D_{pq}^L \le \frac{\hat{u}_P}{\hat{u}_q} \le D_{pq}^U$.

$\delta_1\delta_2(-1)^{\delta_3}\hat{u}_0 = \delta_1\delta_2(-1)^{\delta_3}\tilde{u}_0 \ge 0$,Because ε is a non-Archimedean infinite fraction, so $u_r^{'} \ge \varepsilon$, $v_i^{'} \ge \varepsilon$.

After the above proof, $(u_o^{'},U^{'},V^{'})$ is a feasible solution of model (7), and the corresponding objective function value of the feasible solution is: $\sum_{r=1}^{s}\hat{u}_r y_{rj_0} - \delta_1\hat{u}_0 = \sum_{r=1}^{s}\tilde{u}_k \frac{\sum_{r=1}^{s}\tilde{u}_r (Y_{rj_o})_\alpha^L}{\sum_{r=1}^{s}\tilde{u}_r y_{rj_0}} y_{rj_0} - \delta_1\tilde{u}_0 = \sum_{r=1}^{s}\tilde{u}_r (Y_{rj_o})_\alpha^L - \delta_1\tilde{u}_0 = (E_{j_o})_\alpha^L$.So there is $(E_{j_0})_\alpha^L \le E_{j_0}^\alpha$, that is, if the input and output factors come from $(\tilde{X}_{ij})_\alpha$ and $(\tilde{Y}_{ij})_\alpha$, then the model (8) can obtain the minimum efficiency value. It should be noted that the model (4) is the minimum value that can be obtained at the level of α -cut set. It can be seen that the model (4) and (8) will obtain the same optimal value at the level of α, so the solution of the two models is the same, and theorem 2 is proved.

Through the proof of theorem 1 and theorem 2, the optimal value of the upper and lower limit of the efficiency value of the α decision-making unit under the α level can be obtained by equation (6) and model (8).

(2) Simplified calculation of α -cut set

Although equations (6) and (8) are linear programming, their calculationis limited by level α, so the calculation procedure of efficiency value is complex. To simplify the procedure, this paper will use triangular fuzzy numbers to represent $(\tilde{X}_{ij})_\alpha$ and $(\tilde{Y}_{ij})_\alpha$.

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First, triangular fuzzy numbers are used to define $\tilde{X}_{ij}$ and $\tilde{Y}_{ij}$ as follows:

Definition 2: The triangular fuzzy function can be defined as, and its membership function can be expressed as:

$$\mu(x)=\begin{cases}0 & x\le x^1 \quad or \quad x\ge x^3\\ \dfrac{x-x^1}{x^2-x^1} & x^1\le x\le x^2\\ \dfrac{x^3-x}{x^3-x^2} & x^2\le x\le x^3\end{cases}$$

According to definition 2, $\tilde{X}_{ij}$ and $\tilde{Y}_{ij}$ can be represented by triangular fuzzy numbers: $\tilde{X}_{ij}=\left\{x_{ij}^1,x_{ij}^2,x_{ij}^3\right\}$, $\tilde{Y}_{ij}=\left\{y_{rj}^1,y_{rj}^2,y_{rj}^3\right\}$. Use any α -cut set to limit $\tilde{X}_{ij}$ and $\tilde{Y}_{ij}$, and the corresponding upper and lower bounds of the membershipfunction can be expressed as:

$$(X_{ij})_\alpha^L=x_{ij}^1+\alpha(x_{ij}^2-x_{ij}^1);(X_{ij})_\alpha^U=x_{ij}^2+\alpha(x_{ij}^3-x_{ij}^2) \tag{9}$$

$$(Y_{rj})_\alpha^L=y_{rj}^1+\alpha(y_{rj}^2-y_{rj}^1),(Y_{rj})_\alpha^U=y_{rj}^2+\alpha(y_{rj}^3-y_{rj}^2) \tag{10}$$

Bring the above formula into Formula (6) and Formula (8) to get: $(E_{j_o})_\alpha^U=\max\sum_{r=1}^{s}u_r(y_{rj}^2+\alpha(y_{rj}^3-y_{rj}^2))-\delta_1u_o$

$$st.\begin{cases}\sum_{i=1}^{m}v_i(x_{ij}^1+\alpha_i(x_{ij}^2-x_{ij}^1))=1,i=1,2,\ldots,m,\\ \sum_{r=1}^{s}u_r(y_{rj}^4+\alpha_i(y_{rj}^4-y_{rj}^3))-\sum_{i=1}^{m}v_i(x_{ij}^1+\alpha_i(x_{ij}^2-x_{ij}^1))-\delta_1u_o\le 0\\ \sum_{r=1}^{s}u_r(y_{rj}^1+\alpha_i(y_{rj}^2-y_{rj}^1))-\sum_{i=1}^{m}v_i(x_{ij}^3+\alpha_i(x_{ij}^3-x_{ij}^2))-\delta_1u_o\le 0\\ C_{pq}^Lv_q\le v_p\le C_{pq}^Uv_q\\ D_{pq}^Lu_q\le u_p\le D_{pq}^Uu_q\\ v_i\ge\varepsilon,u_r\ge\varepsilon,r=1,\ldots,s,\\ \delta_1\delta_2(-1)^{\delta_3}u_o\ge 0,j=1,2,\ldots,n,j\ne j_o\end{cases} \tag{11}$$

$$(E_{j_o})_\alpha^L=\max\sum_{r=1}^{s}u_r(y_{rj}^1+\alpha(y_{rj}^2-y_{rj}^1))-\delta_1u_o$$

$$st.\begin{cases}\sum_{i=1}^{m}v_i(x_{ij}^2+\alpha(x_{ij}^3-x_{ij}^2))=1,i=1,2,\ldots,m,\\ \sum_{r=1}^{s}u_r(y_{rj}^1+\alpha(y_{rj}^2-y_{rj}^1))-\sum_{i=1}^{m}v_i(x_{ij}^2+\alpha(x_{ij}^3-x_{ij}^2))-\delta_1u_o\le 0,\\ \sum_{r=1}^{s}u_r(y_{rj}^2+\alpha(y_{rj}^3-y_{rj}^2))-\sum_{i=1}^{m}v_i(x_{ij}^1+\alpha(x_{ij}^2-x_{ij}^1))-\delta_1u_o\le 0,\\ C_{pq}^Lv_q\le v_p\le C_{pq}^Uv_q\\ D_{pq}^Lu_q\le u_p\le D_{pq}^Uu_q\\ v_i\ge\varepsilon,u_r\ge\varepsilon,r=1,\ldots,s,\\ \delta_1\delta_2(-1)^{\delta_3}u_o\ge 0,j\ne j_o\end{cases} \tag{12}$$

By transforming $\tilde{X}_{ij}$ and $\tilde{Y}_{ij}$ into triangular fuzzy numbers, the calculation o α -cut set level can be simplified. The efficiency value $((E_{j_o})_\alpha^L,(E_{j_o})_\alpha^U)$ of the generalized fuzzy DEA/AR model is obtained by solving equations (11) and (12).

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3.4. Monte Carlo simulation sequencing of generalized fuzzy DEA/AR

However, as $((E_{j_o})_\alpha^L,(E_{j_o})_\alpha^U)$ is an interval efficiency value, it is difficult to sort it. Moreover, these sorting methods are divorced from the practical production principle of DEA method. In view of this paper will use Monte Carlo simulation method to sort the interval efficiency value $((E_{j_o})_\alpha^L,(E_{j_o})_\alpha^U)$ on the basis of analyzing the effective production principle of DEA method.

First, the effective production principle in DEA model is analyzed as shown in Figure 2.The horizontal axis represents the efficiency value e, and the vertical axis represents the output factor y_{ij}, whose production possibility boundary is curve OGH. Consider a decision-making unit DMU_j. According to equations (6) and (8), its efficiency value is $((E_{j_o})_\alpha^L,(E_{j_o})_\alpha^U)$ and output variable is $((Y_{rj})_\alpha^L,(Y_{rj})_\alpha^U)$. The positions of each variable are shown in Figure 2. The region $ACDE$ formed by the four variables is its effective production region. As can be seen from Figure 2, its effective production region is located around the production possibility boundary OGH. If the production point falls within this region, its production is relatively efficient. If the production point falls outside this area, it is relatively ineffective. According to Figure 2, the effective production area V_j^* of the decision-making unit is:

$$V_j^* = \prod_{j=1, j\neq j_o}^{m} ((E_{j_o})_\alpha^L,(E_{j_o})_\alpha^U) \times \prod_{j=1, j\neq j_o}^{m} ((Y_{rj})_\alpha^L,(Y_{rj})_\alpha^U) \tag{13}$$

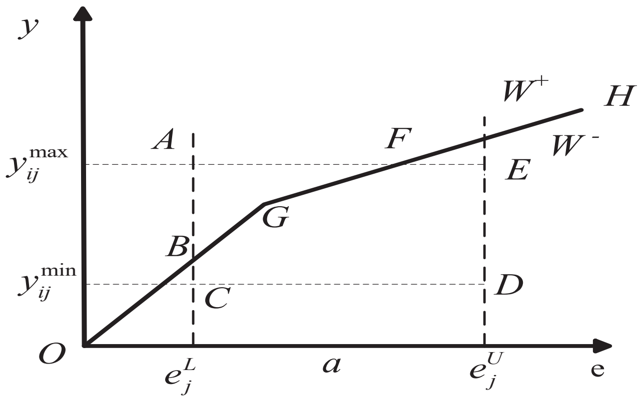

Figure 2. Effective production principle of DEA method

Secondly, the production efficiency of decision unit is simulated according to Monte Carlo simulation method. The efficiency measured by DEA is a ratio, which measures the ratio between The Times of production belonging to the effective production area and The Times of production not belonging to the effective production area. If most production points of a decision unit belong to the effective production area, the efficiency is higher. On the contrary, its efficiency is lower. In this paper, $\bar{W}=(\bar{X}_{ij},\bar{Y}_{rj})$ is used to simulate the location of the production point of decision making unit, $\bar{y}_{rj} \sim U((Y_{rj})_\alpha^L,(Y_{rj})_\alpha^U)$ and $\bar{E}_j \sim U((E_{j_o})_\alpha^L,(E_{j_o})_\alpha^U)$ are set, and the DEA efficiency is represented by the index M_{R_j}, namely:

$$M_{R_j} = \frac{N_{H_j}}{N_{H_j}+N_{M_j}} V_j^* \tag{14}$$

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Where, N_{H_j} is the number of production simulation within the effective production area; N_{M_j} is the number of production simulation outside the effective production area; V_j^* is the area of the effective production area;the calculation method is shown in Equation (13).

4. APPLICATION EXAMPLES

This part will take the public cultural service efficiency evaluation of 13 cities in the Beijing-Tianjin-Hebei City cluster as the research object to verify the validity of the generalized fuzzy DEA/AR model proposed in this paper based on Monte Carlo simulation.

4.1. Index selection and data sources

Firstly, in terms of index selection, literature (1 and 2) proposes that the input factors of public cultural services should include two aspects: public cultural funds and cultural practitioners. In reference (4), the output of public cultural services includes two aspects -- hardware facilities and cultural activities. In this paper, the two aspects of reference 4 are divided into two indicators: the number of public cultural facilities (including museums, libraries and cultural centers at district level and above) and the number of art activities of each group. At the same time, considering that residents' satisfaction is an important index to measure the quality of public cultural services, this paper also takes "residents' satisfaction with public cultural services" as its output index.

Secondly, the quantification of indicators comes from National Data Digital Culture Network, 2015 Statistical Yearbook of each city, China Cultural Relics Statistical Yearbook (2015) and China Social Statistical Yearbook (2015-2016). "Residents' satisfaction with public cultural services" is a fuzzy index. This paper obtains its data through triangular fuzzy logic scoring, and sets the scoring criteria as follows: The corresponding triangular fuzzy numbers of "satisfied", "general" and "unsatisfied" are respectively (0,0.2,0.4), (0.3,0.5, 0.7) and (0.6,0.8,1), and their membership functions are shown in Figure 2.

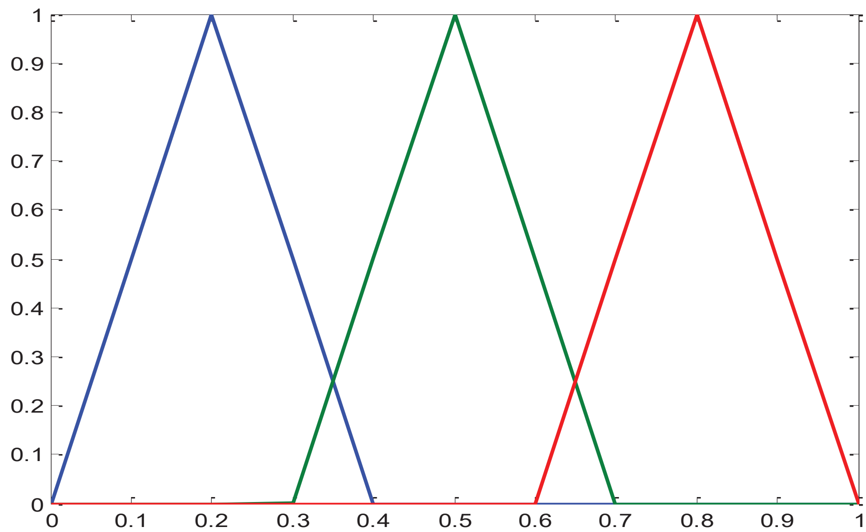

Figure 2. Trigonometric fuzzy number boundary and its membership function

Based on the above analysis, the input-output index and its descriptive statistics are shown in Table 1:

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Table 1. Input-output indicators and descriptive statistics

Index name		Mean value	Standard deviation
Input index	Per capita public cultural funds (ten thousand yuan/person)	0.0528	32.28
	Number of public culture employees (PCS)	427.31	26.62
Output index	Number of public cultural facilities (each)	87.53	43.73
	Residents' satisfaction with public cultural services	(0.25,0.43, 0.72)	23.54
	Number of Art activities per 1000 people (times / 1000 people)	7.2856	9.15

4.2. Empirical analysis results and discussion

Through expert consultation, the weight of input index is estimated as: $v_1 \in (0.72, 0.84)$, $v_2 \in (0.15, 0.23)$; The weight of output index is estimated to be $u_1 \in (0.33, 0.48)$, $u_2 \in (0.21, 0.37)$, $u_3 \in (0.34, 0.53)$. From the comparison of weights of input-output factors, the range of assurance domain (AR) can be obtained as follows:

$$\frac{0.72}{0.15} \le \frac{v_1}{v_2} \le \frac{0.84}{0.23}, \frac{0.33}{0.21} \le \frac{u_1}{u_2} \le \frac{0.48}{0.37}, \frac{0.33}{0.34} \le \frac{u_1}{u_3} \le \frac{0.48}{0.53}, \frac{0.21}{0.34} \le \frac{u_2}{u_3} \le \frac{0.37}{0.53}.$$

Based on the definition of AR scope, this paper firstly uses matlab17.0 to calculate the public cultural service efficiency of 13 cities in the Beijing- Tianjin- Hebei metropolitan area according to equations (11) and (12).Secondly, this paper uses the MATLAB simulation method to simulate the production times for 1000 times, and calculates M_{R_j} by calculating the simulation times N_{H_j} in the effective production area and N_{M_j} outside the effective production area, thus ranking the 13 cities. For different values of δ_1, δ_2 and δ_3, the public cultural service efficiency values of 13 cities and their ranking are shown in Table (2).

Table 2. The calculation results and ranking of public cultural service efficiency in 13 cities under different δ_i levels.

city	$\delta_1 = 0$ (blur CCR-DEA)			$\delta_1 = 1$, $\delta_2 = 0$ (blurBCC-DEA)			$\delta_1 = \delta_2 = 1$, $\delta_3 = 0$ (blurFG-DEA)			$\delta_1 = \delta_2 = \delta_3 = 1$ (blurST-DEA)		
	$((E_{j_o})^L_\alpha, (E_{j_o})^U_\alpha)$	M_{R_j}	Ranking	$((E_{j_o})^L_\alpha, (E_{j_o})^U_\alpha)$	M_{R_j}	Ranking	$((E_{j_o})^L_\alpha, (E_{j_o})^U_\alpha)$	M_{R_j}	Ranking	$((E_{j_o})^L_\alpha, (E_{j_o})^U_\alpha)$	M_{R_j}	Ranking
Beijing	(1,1)	9.73	1	(1,1)	8.05	2	(1,1)	9.06	1	(1,1)	8.84	1
Qinhuangdao	(1,1)	9.44	2	(1,1)	7.44	4	(1,1)	8.11	4	(1,1)	8.16	3
Tianjin	(1,1)	8.36	3	(1,1)	8.11	1	(1,1)	8.72	2	(1,1)	8.32	2
Zhangjiakou	(0.41,0.89)	7.58	4	(0.61,0.73)	6.27	7	(0.57,0.73)	6.34	8	(0.58,0.79)	7.45	5
Baoding	(0.43,0.85)	7.21	5	(0.62,0.85)	7.15	5	(1,1)	8.46	3	(0.51,0.75)	7.22	6
Handan	(0.31,0.62)	6.79	6	(1,1)	7.69	3	(0.61,0.84)	7.29	6	(0.66,0.82)	7.68	4
Xingtai	(0.24,0.43)	6.46	7	(0.52,0.74)	5.59	8	(0.72,0.88)	7.65	5	(0.48,0.69)	6.36	8
Tangshan	(0.18,0.44)	5.28	8	(0.45,0.63)	4.71	10	(0.55,0.79)	6.85	7	(0.41,0.62)	4.22	10
Cangzhou	(0.22,0.38)	4.95	9	(0.48,0.69)	5.18	9	(0.27,0.45)	3.58	13	(0.35,0.52)	3.55	12
Hengshui	(0.15,0.41)	4.53	10	(0.28,0.45)	3.18	13	(0.43,0.58)	5.15	11	(0.43,0.65)	5.19	9
Chengde	(0.23,0.32)	4.09	11	(0.54,0.78)	6.82	6	(0.51,0.69)	6.18	9	(0.57,0.72)	6.71	7
Langfang	(0.16,0.37)	3.22	12	(0.34,0.55)	3.42	12	(0.47,0.63)	5.62	10	(0.33,0.58)	3.92	11
Shijiazhuang	(0.14,0.29)	2.76	13	(0.39,0.58)	4.26	11	(0.36,0.52)	4.37	12	(0.29,0.48)	3.26	13

Note: Under different δ_i levels, the generalized fuzzy DEA/AR model constructed in this paper can be transformed into fuzzy CCR-DEA model, etc., which indicates that the model has wide applicability, which is the connotation of the "generalized" of the model.

The calculation results in Table 2 show that:

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First of all, the generalized fuzzy DEA/AR model constructed in this paper has wide applicability. According to different guarantee domains (different δ_i levels), the generalized fuzzy DEA/AR model can be CCR-DEA model, BCC-DEA model, FFG-DEA model and FST-DEA model. Thus, the efficiency measurement of decision units with fuzzy input-output factors can be realized, which provides a new calculation method for DEA model calculation. The second, with the help of Monte Carlo simulation method, generalized fuzzy DEA/AR model can realize the ranking of interval efficiency values, such as: under the FCCR-DEA model, Beijing, Qinhuangdao and Tianjin are all effective production units, which cannot be sorted by the traditional DEA model. By virtue of the effective production principle of DEA model and the M_{R_j} coefficient constructed by Monte Carlo simulation in this paper, the effective production units can be sorted, which improves the effectiveness of the DEA model sorting method.

Secondly, from the perspective of public cultural service efficiency of 13 cities in the Beijing-Tianjin-Hebei City cluster: First, at different δ_i levels, Beijing, Tianjin and Qinhuangdao all rank in the top three (except for the FFG-DEA model, Baoding ranks the third and Qinhuangdao ranks the fourth) (see Figure 3). Combined with the efficiency measurement results in Table 2, the efficiency values of these three cities are all 1. It can be seen that, the data frontier of the efficiency measurement of 13 public cultural services in the Beijing- Tianjin- Hebei city cluster is determined by these three cities. Second, there is a large difference in the efficiency of public cultural services among the 13 cities in the Beijing-Tianjin-Hebei urban agglomeration. The difference in the efficiency of public cultural services of the highest city is 1, while that of the lowest city is only 0.14 (Baoding City under the FCCR-DEA model).

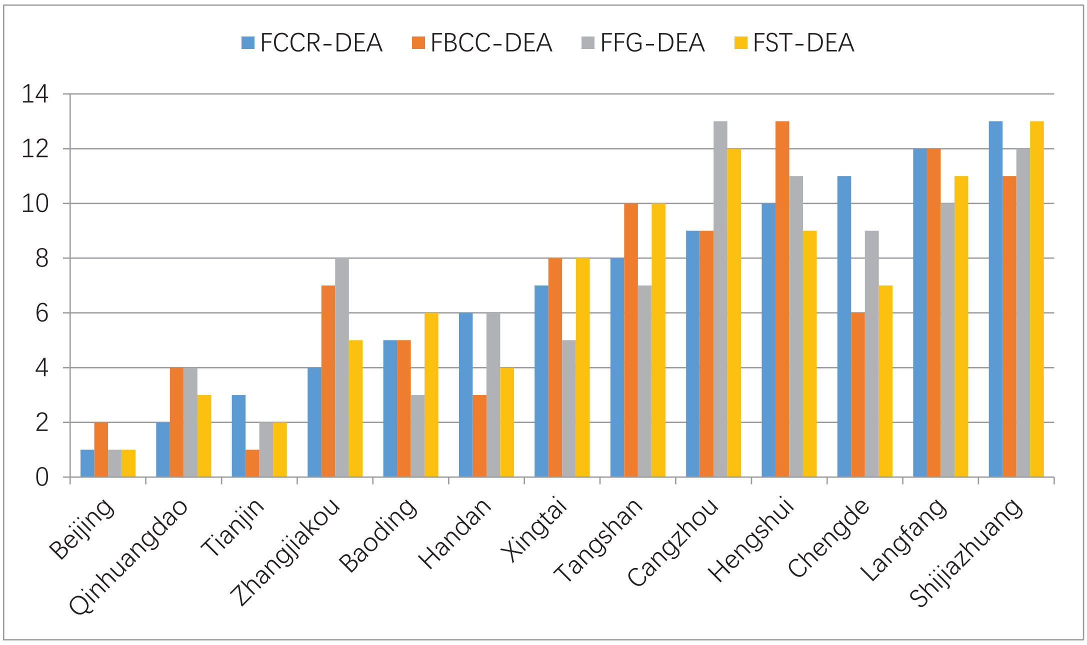

Figure 3. Ranking of public cultural service efficiency of 13 cities at different levels

5. CONCLUSIONS

In this paper, the traditional DEA model is improved and the generalized addition fuzzy is constructed. A new two-stage Stackelberg fuzzy DEA model is proposed to calculate the efficiency values of DMUs and sub-DMUs. The model is linear, which makes it possible to quickly obtain the global optimal solution. And the model is independent of the α cut set, which greatly reduces the calculation amount of the model and improves the accuracy of efficiency calculation. In order to rank the DMUs, this paper uses model Carlo simulation to rank the decision-making units. Finally, the efficiency of public cultural services in 13 cities of Beijing-Tianjin-Hebei urban agglomeration is calculated to verify the effectiveness of the generalized fuzzy DEA/AR model.

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