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Numerical Solution of Fuzzy Nonlinear Volterra-fredholm Integral Equation

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Abstract

In this paper, the uniqueness of the solution of fuzzy nonlinear Volterra-Fredholm integral equation is proved by using Banach fixed point theorem. Next, two modified Adomian decomposition methods (MADM) and standard Adomian decomposition method (SADM) are used to solve the equation numerically. Finally, some numerical examples show that the MADM corresponding to the decomposition of the source function into two parts is more accurate than the SADM and the MADM corresponding to the decomposition of the source function into infinite series.

Keywords

Fuzzy integral equations; Volterra-Fredholm integral equations; Fixed point theorem; Adomian decomposition methods.

1. INTRODUCTION

The integral equation first appeared in a mechanical problem studied by Abel [1], which solved the time required for a particle to slide from any point on a given smooth curve to the bottom under the action of gravity. With the development of computer science and technology, the application range of integral equation is more and more extensive. Many problems in applied disciplines can be transformed into integral equations, such as heat conduction, rotational elasticity, light scattering, forced convection and diffraction. But in the mathematical modeling of such practical problems, the parameters are often uncertain. Therefore, fuzzy integral equations, as well as fuzzy differential equations and fuzzy integro-differential equations, with fuzzy sets as parameters are proposed to solve this problem.

Fuzzy integral equations are closely related to the field of fuzzy control. There are two important aspects in the study of fuzzy integral equations. One is to find different numerical methods to solve the fuzzy integral equations. Such as Adomian decomposition method [2], homotropy analysis method [3–5], successive iterative method using hybrid functions [6] and some modified methods [7–9]. These methods are also extended to fuzzy integro-differential equations [10–14] and integro-differential equations [15]. In these numerical methods, the Adomian decomposition method is an effective method for solving both linear and nonlinear problems [16–18]. It can produce fast convergent solution sequences and is used for deterministic and stochastic equations.

The other is the theoretical study of existence, uniqueness for fuzzy integral equations. Let E n be the n-dimensional fuzzy number space, which will be introduced in Definition 2.1. Given a fuzzy Volterra-Fredholm integral equations

$$x(t) = F\left(t, x(t), \int_0^t f\left(t, s, x(s)\right)\mathrm{d}s, \int_0^T g\left(t, s, x(s)\right)\mathrm{d}s\right), 0 \le t \le T, \tag{1.1}$$

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where $x(t): [0, T] \to E^n$ is an unknown fuzzy function, $f, g \in C(\triangle\times E^n, E^n)$ with $\triangle=\{(t,s): 0 \le s \le T\}$, $F \in C([0,T] \times E^n \times E^n \times E^n, E^n)$. Park and Jeong [19] established the existence and uniqueness of the solutions of Eq.(1.2).

Next, Balachandran and Prakash [20] extended the results in [19] to the general nonlinear form

$$x(t) = F(t, x(t), \int_0^t f_1\left(t,s,x(s)\right)\mathrm{d}s, \cdots, f_m\left(t,s,x(s)\right)\mathrm{d}s, \int_0^T g_1\left(t,s,x(s)\right)\mathrm{d}s, \cdots, \int_0^T g_m\left(t,s,x(s)\right)\mathrm{d}s), 0 \le t \le T, \tag{1.2}$$

where $x: [0, T] \to E^n$is an unknown fuzzy set-valued mapping, $F \in C\left([0,T] \times E^{(2m+1)n}, E^n\right)$.

In 2011, Hajighasemi [21] investigated the existence and uniqueness of solution of the fuzzy Volterra-Fredholm integral equation

$$x(t) = f(t) + \int_0^t k\,(t,s) g\left(s, x(s)\right)\mathrm{d}s + \int_0^a h\,(t,s) q\left(s, x(s)\right)\mathrm{d}s, 0 \le t \le a, \tag{1.3}$$

where $x(t)$ is an unknown fuzzy set-valued mapping, $f \in C([0,a], E)$, $k, h \in C(\triangle, E)$ with $\triangle=\{(t,s): 0 \le s \le t \le a\}$, $g, q \in C([0,a] \times E, E)$.

In 2012, Khezerloo and Hajighasemi [22] studied the existence and uniqueness of solutions for the fuzzy Volterra integral equations given by

$$x(t) = f(t) + \int_0^t K\,(t,s) g\left(x, x(s)\right)\mathrm{d}s, t \ge 0, \tag{1.4}$$

where $x(t)$ is an unknown fuzzy set-valued mapping, $f \in C([0,a], E)$, $K \in C([0,a] \times [0,a], E)$, $g \in C([0,a] \times E, E)$.

In this paper, we will focus on the following fuzzy nonlinear Voltera-Fredholm integral equation

$$x(t) = f(t) + \int_0^t K_1\,(t,s) F_1\left(s, x(s)\right)\mathrm{d}s + \int_0^1 K_2\,(t,s) F_2\left(s, x(s)\right)\mathrm{d}s, 0 \le t \le 1, \tag{1.5}$$

where $f \in C([0,1], E)$, $K_1 \in C(\triangle, E)$ with $\triangle=\{(t,s): 0 \le s \le t \le 1\}$, $K_2 \in C([0,1] \times [0,1], E)$, nonlinear functions $F_i \in C([0,1] \times E, E) (i = 1,2)$ need to satisfy some conditions which will be stated later.

Inspired by the above literature, we will discuss the case where the functions g and q in Eq.(1.3) are nonlinear, denoted by F1 and F2 in Eq.(1.5). The Adomian polynomial is used to approximate the nonlinear term, and the differences between the two MADM and SADM in terms of accuracy and computational complexity are compared. In addition, the uniqueness of the solution of Eq.(1.5) is proved by using Banach fixed point theorem.

This work is organized as follows: Section 2 briefly introduces the concept of fuzzy correlation. Section 3 introduces SADM and two MADM. In Section 4, the uniqueness result of the solution of the equation is given. Section 5 provides two numerical examples to show the effectiveness of the method. Finally, Section 6 gives a brief conclusion.

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2. PRELIMINARIES

In this section, we briefly introduce some concepts of fuzzy theory. More contents about fuzzy theory are shown in [24–28].

Definition 2.1. [25] Let $\mathcal{F}(R)$ be the set of all fuzzy sets on R. Let $u \in \mathcal{F}(R)$, if u satisfies

(i) u is a normal fuzzy set, i.e., there exists $x_0 \in R$ such that $u(x_0) = 1$, (ii) u is a convex fuzzy set, i.e., $u(\lambda x_1 + (1-\lambda)x_2) \geq \min\{u(x_1), u(x_2)\}$ for all $x_1, x_2 \in R$ and $\lambda \in [0,1]$, (iii) u is an upper semi-continuous function, (iv) The closure of the support of u is compact, i.e., $[u]^0$ is compact, then u is called as a fuzzy number. The set of all n-dimensional fuzzy numbers is known as the n-dimensional fuzzy number space, denoted by E^n. Let E be a one-dimensional fuzzy number space.

Definition 2.2. [14] Given $0 \leq r \leq 1$, a fuzzy number u in parametric form is represented by an ordered function pairs $\left(\underline{u}(r), \overline{u}(r)\right)$ satisfying

(i) $\underline{u}(r)$ is a bounded left continuous non decreasing function,

(ii) $\overline{u}(r)$ is a bounded left continuous non increasing function,

(iii) $\underline{u}(r) \leq \overline{u}(r)$.

For $u = (\underline{u}, \overline{u}), v = (\underline{v}, \overline{v}) \in E$ and $\lambda \in R$, the sum of $u+v$ and the scalar multiplication λu can be defined by

$$\left(\underline{u+v}\right)(r) = \underline{u}(r) + \underline{v}(r), \quad \left(\overline{u+v}\right)(r) = \overline{u}(r) + \overline{v}(r), \quad \forall r \in [0,1],$$

and

$$\lambda u = \begin{cases} \left(\lambda \underline{u}, \lambda \overline{u}\right), \lambda \geq 0, \\ \left(\lambda \overline{u}, \lambda \underline{u}\right), \lambda \leq 0. \end{cases}$$

Definition 2.3.[23] For any two fuzzy numbers u and v, define $D: E \times E \to R^+ \cup \{0\}$ by

$$D(u,v) = \sup_{r \in [0,1]} \max\{|\ \underline{u}(r) - \underline{v}(r)\ |, |\ \overline{u}(r) - \overline{v}(r)\ |\},$$

where $\mathrm{u} = [\underline{\mathrm{u}}(\mathrm{r}), \overline{\mathrm{u}}(\mathrm{r})], \mathrm{v} = [\underline{\mathrm{v}}(\mathrm{r}), \overline{\mathrm{v}}(\mathrm{r})]$. It concludes that D is a metric on E and (E, D) is a complete metric space with the following useful properties [26].

For $\forall u, v, w, \alpha \in E$ and $\forall \lambda \in R$, there are

(i) $D(u+w, v+w) = D(u,v)$,

(ii) $D(u+v, w+z) \leq D(u,w) + D(v,z)$,

(iii) $D(u,v) \leq D(u,w) + D(w,v)$,

(iv) $D\left(\int_I f(t)\mathrm{d}t, \int_I g(t)\mathrm{d}t\right) \leq \int_I D\left(f(t), g(t)\right)\mathrm{d}t$,

(v) $D\left(u \tilde{*} v, \tilde{0}\right) = D\left(u, \tilde{0}\right) D\left(v, \tilde{0}\right)$ with the fuzzy multiplication $\tilde{*}$ is based on the extension principle that can be proved by α -cuts of fuzzy numbers $u, v \in E$. Here $\tilde{0} \in E$ is defined by (see [27])

$$\tilde{0}(x) = \begin{cases} 1, x = 0, \\ 0, elsewhere. \end{cases}$$

(vi) $D(\lambda_1 u, \lambda_2 u) = |\ \lambda_1 - \lambda_2\ |\ D\left(u, \tilde{0}\right)$ with $\lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1 \lambda_2 > 0$ (see [28]).

3. ADOMIAN DECOMPOSITION METHOD

The solution $x(t)$ of Eq.(1.5) can be expressed by the infinite series

$$x(t) = \sum_{i=0}^{\infty} x_i\,(t) \tag{3.1}$$

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The Adomian decomposition method identifies the nonlinear terms F_1 and F_2 by the decomposing series

$$F_1\big(t,x(t)\big)=\sum_{n=0}^{\infty}A_n(t),\quad F_2\big(t,x(t)\big)=\sum_{n=0}^{\infty}B_n(t), \tag{3.2}$$

with the Adomian polynomials A_n and B_n are given by

$$A_n=\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[F_1\left(\sum_{i=0}^{n}\lambda^i x_i\right)\right]_{\lambda=0}, n=0,1,2,\cdots.$$

$$B_n=\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[F_2\left(\sum_{i=0}^{n}\lambda^i x_i\right)\right]_{\lambda=0}, n=0,1,2,\cdots$$

Substitute Eqs.(3.1)-(3.2) into Eq.(1.5) to obtain

$$\sum_{i=0}^{\infty}x_i(t)=f(t)+\int_0^t K_1(t,s)\sum_{n=0}^{\infty}A_n(s)\mathrm{d}s+\int_0^1 K_2(t,s)\sum_{n=0}^{\infty}B_n(s)\mathrm{d}s. \tag{3.3}$$

The iteration of SADM [30] is as follows

$$\begin{aligned}x_0(t)&=f(t),\\ x_{i+1}(t)&=\int_0^t K_1(t,s)A_i\mathrm{d}s+\int_0^1 K_2(t,s)B_i\mathrm{d}s, i=0,1,\cdots\end{aligned}$$

In 1999, Wazwaz [30] proposed a MADM on the SADM, which is based on the function $f(t)$ can be decomposed into two parts $f_1(t)$ and $f_2(t)$, That is

$$f(t)=f_1(t)+f_2(t) \tag{3.4}$$

Combining Eq.(3.3) and Eq.(3.4) yields the following iteration

$$\begin{aligned}x_0(t)&=f_1(t),\\ x_1(t)&=f_2(t)+\int_0^t K_1(t,s)A_0\mathrm{d}s+\int_0^1 K_2(t,s)B_0\mathrm{d}s,\\ x_{i+1}(t)&=\int_0^t K_1(t,s)A_i\mathrm{d}s+\int_0^1 K_2(t,s)B_i\mathrm{d}s, i=2,3\cdots\end{aligned}$$

In 2001, Wazwaz and El-Sayed [31] proposed a MADM based on the previous research. This method represents the function $f(t)$ by Taylor series as

$$f(t)=\sum_{i=0}^{\infty}f_i(t) \tag{3.5}$$

The following new iteration can be obtained from Eq.(3.3) and Eq.(3.5)

$$\begin{aligned}x_0(t)&=f_1(t),\\ x_{i+1}(t)&=f_i(t)+\int_0^t K_1(t,s)A_i\mathrm{d}s+\int_0^1 K_2(t,s)B_i\mathrm{d}s, i=1,2\cdots.\end{aligned}$$

4. UNIQUENESS OF THE SOLUTION

Based on the following conditions, the existence and uniqueness of the solution of Eq.(1.5) is proved by the Banach fixed point theorem.

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(A1) $f: I \to E$ is a continuous fuzzy valued function.

(A2) $F_1: I \times E \to E, F_2: I \times E \to E$ are continuous functions and satisfy the Lipschitz condition, i.e., there exist $L_1 > 0$ and $L_2 > 0$ such that

$$D\left(F_1\left(s, x_1(t)\right), F_1\left(s, x_2(t)\right)\right) \leq L_1 D\left(x_1(t), x_2(t)\right),$$

and

$$D\left(F_2\left(s, x_1(t)\right), F_2\left(s, x_2(t)\right)\right) \leq L_2 D\left(x_1(t), x_2(t)\right).$$

(A3) K_1 and K_2 are continuous function and there exist $M_1 > 0, M_2 > 0$ satisfying

$$\int_0^t D\left(K_1(t,s), \tilde{0}\right) \leq M_1, \quad \int_0^1 D\left(K_2(t,s), \tilde{0}\right) \leq M_2,$$

with $0 < \delta := M_1 L_1 + M_2 L_2 < 1$.

Theorem 4.1. Assume that the hypotheses $(A1), (A2)$ and $(A3)$ hold, then Eq.(1.5) has a unique solution $x(t)$.

Proof. Let $T: C([0,1], E) \to C([0,1], E)$ be an operator defined as

$$(Tx)(t) = f(t) + \int_0^t K_1(t,s) F_1\left(s, x(s)\right) \mathrm{d}s + \int_0^1 K_2(t,s) F_2\left(s, x(s)\right) \mathrm{d}s.$$

Then for $\forall y_1, y_2 \in E$ and $\forall t \in I$, there are

$$(Ty_i)(t) = f(t) + \int_0^t K_1(t,s) F_1(s, y_i) \mathrm{d}s + \int_0^1 K_2(t,s) F_2(s, y_i) \mathrm{d}s, (i = 1,2).$$

From the Definition 3, there is

$$\begin{aligned}
& D\left(Ty_1(t), Ty_2(t)\right) \\
\leq\ & D\left(\int_0^t K_1(t,s) F_1(s, y_1) \mathrm{d}s, \int_0^t K_1(t,s) F_1(s, y_2) \mathrm{d}s\right) \\
& + D\left(\int_0^1 K_2(t,s) F_2(s, y_1) \mathrm{d}s, \int_0^1 K_2(t,s) F_2(s, y_2) \mathrm{d}s\right) \\
\leq\ & \int_0^t D\left(K_1(t,s), \tilde{0}\right) D\left(F_1\left(s, y_1(s)\right), F_1\left(s, y_2(s)\right)\right) \mathrm{d}s \\
& + \int_0^1 D\left(K_2(t,s), \tilde{0}\right) D\left(F_2\left(s, y_1(s)\right), F_2\left(s, y_2(s)\right)\right) \mathrm{d}s \\
\leq\ & \sup_{0 \leq s \leq 1} D\left(F_1\left(s, y_1(s)\right), F_1\left(s, y_2(s)\right)\right) \int_0^t D\left(K_1(t,s), \tilde{0}\right) \mathrm{d}s \\
& + \sup_{0 \leq s \leq 1} D\left(F_2\left(s, y_1(s)\right), F_2\left(s, y_2(s)\right)\right) \int_0^1 D\left(K_2(t,s), \tilde{0}\right) \mathrm{d}s \\
\leq\ & (M_1 L_1 + M_2 L_2) D\left(y_1(s), y_2(s)\right).
\end{aligned}$$

Eq.(4.2) means that T is a contraction map. It is concluded that T has a unique fixed point $x(t)$ from the Banach fixed point theorem.

5. NUMERICAL EXAMPLES

In this section, the Adomian polynomial corresponding to the nonlinear term is shown in , and the iterative sequence corresponding to the approximate solution of the equation is referred to in section 3.

Example1 Consider the fuzzy nonlinear Volterra-Fredholm integral equations as

$$\begin{cases} \underline{x}(t) = -\frac{r^2}{300}t^4 - \frac{r^2}{400}t + \frac{r}{10}t + \int_0^t t\,\underline{x}(s)^2 \mathrm{d}s + \int_0^1 t\,s\underline{x}(s)^2 \mathrm{d}s, \ 0 \le s \le t \le 1, \\ \overline{x}(t) = -\frac{(r-2)^2}{300}t^4 - \left(\frac{r}{10} - \frac{1}{5}\right)t - \frac{(r-2)^2}{400}t + \int_0^t t\,\overline{x}(s)^2 \mathrm{d}s + \int_0^1 t\,s\overline{x}(s)^2 \mathrm{d}s \end{cases} \quad (5.1)$$

The exact solution is $x(t) = [\underline{x}(t,r), \overline{x}(t,r)] = \left[\frac{r}{10}t, \frac{2-r}{10}t\right], \ 0 \le r \le 1.$

The left bound errors and the right bound errors are list in Table 1 and Table 2 for $n = 10$, respectively.

Table 1. Left bound of error (when $r = 0.5, n = 4$)

t	E_{SADM}	$E_{MADM(1999)}$	$E_{MADM(2001)}$
0	0	0	0
0.2	8.3093e-16	1.8908e-16	4.7137e-09
0.4	1.7278e-15	3.9205e-16	9.6844e-09
0.6	2.8796e-15	6.5919e-16	1.5782e-08
0.8	4.8225e-15	1.1172e-15	2.5780e-08
1.0	9.2426e-15	2.1025e-15	5.1689e-08

Table 2. Right bound of error (when $r = 0.5, n = 4$)

t	E_{SADM}	$E_{MADM(1999)}$	$E_{MADM(2001)}$
0	0	0	0
0.2	1.6102e-11	3.9477e-12	4.2709e-07
0.4	3.3484e-11	8.2333e-12	8.8134e-07
0.6	5.6007e-11	1.3864e-11	1.4506e-06
0.8	9.3720e-11	2.3354e-11	2.3933e-06
1.0	1.7930e-10	4.4138e-11	4.7110e-06

Example2 Solve the following fuzzy nonlinear Volterra-Fredholm integral equation

$$\begin{cases} \underline{x}(t) = -\frac{r}{6}t - \frac{r^2}{144}t^5 + \frac{r^3}{864}(t^2-1) + \int_0^t t\,s\underline{x}(s)^2 \mathrm{d}s + \int_0^1 (1-t^2)\,\underline{x}(s)^3 \mathrm{d}s, \ 0 \le s \le t \le 1, \\ \overline{x}(t) = -\frac{r-2}{6}t - \frac{(r-2)^2}{144}t^5 - \frac{(r-2)^3}{864}(t^2-1) + \int_0^t t\,s\overline{x}(s)^2 \mathrm{d}s + \int_0^1 (1-t^2)\,\overline{x}(s)^3 \mathrm{d}s. \end{cases} \quad (5\text{-}2)$$

The exact solution is $x(t) = [\underline{x}(t,r), \overline{x}(t,r)] = \left[\frac{r}{6}t, \frac{2-r}{6}t\right], \ 0 \le r \le 1.$

Choose $r = 0.5$ in Eq.(5.2). The left bound errors and the right bound errors are list in Table 3 and Table 4 for $n = 4$.

Table 3. Left bound of error (when $r = 0.5, n = 4$)

t	E_{SADM}	$E_{MADM(1999)}$	$E_{MADM(2001)}$
0	2.9172e-11	9.9033e-13	1.0045e-06
0.2	2.8128e-11	9.5739e-13	4.2128e-07
0.4	2.6050e-11	9.1664e-13	1.6748e-05
0.6	2.5781e-11	1.0367e-12	1.3354e-04
0.8	3.5918e-11	1.9583e-12	5.6646e-04
1.0	1.7825e-10	9.3628e-12	1.7326e-03

Table 4. Right bound of error (when $r = 0.5, n = 4$)

t	E_{SADM}	$E_{MADM(1999)}$	$E_{MADM(2001)}$
0	1.4430e-06	2.4656e-07	2.4318e-04
0.2	1.3913e-06	2.3799e-07	2.2940e-04
0.4	1.2778e-06	2.2128e-07	5.8890e-05
0.6	1.1958e-06	2.1721e-07	9.9350e-04
0.8	1.3305e-06	2.6890e-07	4.8597e-03
1.0	2.6568e-06	5.4505e-07	1.5303e-02

From the results of Table 1-Table 4, it is easy to obtain that the MADM which decomposes the source function into two parts improves the accuracy compared with SADM, and reduces the amount of calculation. Compared with SADM, MADM, which decomposes the source function into series, minimizes the amount of calculation, but also reduces the accuracy.

6. CONCLUSIONS

In this paper, the one-dimensional fuzzy number set is regarded as a closed convex dimension of Banach space, and the existence and uniqueness of solutions of nonlinear fuzzy Volterra-Fredholm equations are analyzed by using abstract theory. The nonlinear part of the equation is approximated by Adomian polynomials. By analyzing the results of several numerical examples, it is found that under the premise that the source term can be decomposed, the MADM obtained by decomposing the source term function into two terms has higher accuracy and better effect than SADM. Although the MADM obtained by series expansion of the source function greatly reduces the amount of calculation, it also reduces the accuracy compared with SADM.

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