Application of the Atangana-Baleanu Fractional Derivative to a Tumor Model

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Abstract

Recently, the fractional derivative with nonsingular and nonlocal kernels in the sense of Caputo and Riemann-Liouville, called Atangana-Baleanu fractional derivatives, had been extensively applied in various fields. And to approximate the fractional derivative more efficiently, a new numerical method based on Lagrange polynomial interpolation was proposed. In this paper, the Atangana-Baleanu fractional derivative was applied to a tumor growth model with strong Allee effect. The approximate solution was obtained through the new numerical method and plotted.

Keywords

Tumor model; Atangana-Baleanu fractional derivative; Numerical method; Practical application.

1. INTRODUCTION

Fractional derivative is an extension of integer derivative. With the intensive research of fractional derivative, it has been found to be a powerful tool to solve complex practical problems with memorability, heredity and nonlocality and has been widely used in many fields such as physics, engineering, finance and economics, and biology [1-7]. There are many forms of fractional differential operators, such as Riemann-Liouville, Caputo, Riesz, Weyl, etc. [8-11]. These differential operators which are all related in some way have their own specific features and properties. Different appropriate representations are usually given depending on the angle of the question being studied. New fractional differential operators are constantly being explored and used to describe real phenomena. In 2016, Atangana and Baleanu promoted a new definition of fractional order differential with non-singular and nonlocal kernel [12]. The Laplace transform and fractional order integral corresponding to the newly promoted differential were also given. There were many methods for numerically solving fractional order differential equations, such as difference method [16], prediction correction method [17] and Adomian decomposition method [18]. In order to solve nonlinear and nonautonomous fractional derivative equations with the Atangana-Baleanu derivative more efficiently, a new numerical method which involves the Lagrange interpolating polynomial was proposed [13].

Fractional order differentiation is an important tool for medical research. Tumor is a genetic disease with many complex characteristics such as spread, metastasis, and mutation. Therefore, in the literatures of recent years, fractional orders have been introduced in tumor models and then kinetic studies have been carried out. In 2012, Rihan and Fathalla proposed a fractional-order cancer model to obtain a stable memory state of the immune system [19]. In 2014, Bolton et al [20] presented the fractional-order Gompertz model. The investigation results showed that the 0.68-order fractional Gompertz model was more in line with their experimental datas set than the well-known ordinary Gompertz model. In 2018, Celik et al [21] proposed the Caputo-Fabrizio fractional order cancer tumor model and explored the uniqueness and existence of its

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solution. In 2020, Carlos et al [22] carried out numerical simulations in the form of fractional and integer orders on the five classical tumor growth models, and the results showed that the fractional order models had better performance. In 2022, Khajanchi et al [15] described a tumor growth model with Allee effects using the Caputo-Fabrizio fractional order and acquired the implicit form of the analytical solution. Inspired by the above literatures and recognizing the paucity of literature on the use of Atangana-Baleanu fractional derivatives for modeling tumor growth, in this article the Atangana-Baleanu fractional order in the sence of Caputo was considered to describe a tumor model from [15]. The approximate solutions was obtained by using Lagrange interpolating polynomial.

This paper is structured as follows: in Section 2, the definitions and properties associated with the Atangana-Baleanu-Caputo (ABC) fractional order differentiation are reviewed; in Section 3, the tumor model depicted in the Atangana-Baleanu-Caputo fractional order is solved; and the last section is a brief summary.

2. PRELIMINARIES

In this section, we recall some basic definitions and properties of ABC derivative. And the fractional tumor model that is investigated in next section is introduced. The Sobolev space of order one is defined by

$$H^1(a,b) = \{u \in L^1(a,b) : u' \in L^2(a,b)\}.$$

2.1. The ABC fractional derivative

Definition 2.1.1[12]. The Atangana-Baleanu fractional derivative of order $\alpha \in [0,1)$ in Caputo sense of the function $f(t) \in H^1(a,b)$ is defined as follows

$$ {}^{ABC}_{\ \ \alpha}Df(t) = \frac{N(\alpha)}{1-\alpha}\int_a^t E_\alpha[-U_\alpha(t-x)^\alpha]f'(x)dx, \tag{2.1.1}$$

where $N(\alpha)$ is a normalization function obeying $N(0) = N(1) = 1, U_\alpha = \frac{\alpha}{1-\alpha}$ and $E_\alpha(t) = \sum_{k=0}^{+\infty}\frac{t^k}{\Gamma(k\alpha+1)}$ is the Mittag-Leffler function of parameter α.

Definition 2.1.2[12]. The extended fractional integral operator corresponding to ${}^{ABC}_{\ \ \alpha}D$ is defined by

$$ {}^{ABC}_{\ \ \alpha}If(t) = \frac{1-\alpha}{N(\alpha)}f(t) + \frac{\alpha}{N(\alpha)}\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau. \tag{2.1.2}$$

Theorem 2.1.1[14]. Let $f(t) \in H^1(a,b), \alpha \in [0,1), \beta > 0,$ there are the following properties by extending the Newton-Leibniz formula

$$ {}^{ABC}_{\ \ \alpha}I({}^{ABC}_{\ \ \alpha}Df)(t) = f(t) - f(a), \tag{2.1.3}$$

$$ {}^{ABC}_{\ \ \alpha}D({}^{ABC}_{\ \ \alpha}If)(t) = f(t) - f(a). \tag{2.1.4}$$

2.2. The tumor model

The following form of tumor models with strong Allee effects from [15] is explored in this paper

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$$\frac{dT}{dt} = \lambda T\left(1 - \frac{T}{k}\right)(T - c). \tag{2.2.1}$$

We suppose that the growth of tumor cells is slowed down owing to nutrient deficiency and that the original population size of the tumor is $T(0) = T_0 \geq 0$. Among them, λ and k represent the intrinsic growth rate and the maximum carrying capacity of the tumor cells, respectively. The parameter $c \geq 0$ represents the strong Allee threshold. Affected by the Allee effect, the growth of cell populations decreases at low densities, which is related to the visibility threshold size of that population. The long-term persistence of populations is potentially altered. Notice that for $T \in [0, k]$, T is increasing when $T > c$, T is decreasing when $T < c$.

Fractional order differentiation now has good practical uses in various fields such as medicine, biology and physics, solving many complex real-world problems. This provides a good mathematical foundation for solving the corresponding problems. There are many forms of fractional differential operators. Depending on the characteristics of the tumor, we choose the Atangana-Baleanu-Caputo fractional order operator to apply to the tumor model (2.2.1) and then obtain the following fractional order differential equation to be solved next

$${}^{ABC}_{\alpha}DT(t) = \lambda T\left(1 - \frac{T}{k}\right)(T - c), \tag{2.2.2}$$

where $\alpha \in (0,1)$, ${}^{ABC}_{\alpha}D$ represents the Atangana-Baleanu-Caputo fractional derivative.

3. NUMERICAL SOLUTION

In this section, we solve the fractional order equation (2.2.2) with a new numerical method based on Lagrange interpolation polynomials. And the results of different fractional orders are shown by the image.

For convenience, let $q(t, T(t)) = \lambda T\left(1 - \frac{T}{k}\right)(T - c)$, then the equation (2.2.2) is simplified as

$${}^{ABC}_{\alpha}DT(t) = q(t, T(t)).$$

Integrating both sides of above equation yields

$${}^{ABC}_{\alpha}I({}^{ABC}_{\alpha}DT)(t) = {}^{ABC}_{\alpha}Iq(t, T(t)).$$

From equation (2.1.3) and equation (2.1.2), we can obtain

$$T(t) - T(0) = \frac{1-\alpha}{N(\alpha)}q(t, T(t)) + \frac{\alpha}{N(\alpha)}\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}q(\tau, T(\tau))d\tau.$$

Let $t_n = nh$, $h = t_i - t_{i-1}, n \in N$. The initial value $T(0) = T_0 \geq 0$. We can get

$$T(t_{n+1}) = T_0 + \frac{1-\alpha}{N(\alpha)}q\big(t_n, T(t_n)\big) + \frac{\alpha}{N(\alpha)}\frac{1}{\Gamma(\alpha)}\int_0^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1}q(\tau, T(\tau))d\tau,$$

which leads to

$$T(t_{n+1}) = T_0 + \frac{1-\alpha}{N(\alpha)}q\big(t_n, T(t_n)\big) + \frac{\alpha}{N(\alpha)}\frac{1}{\Gamma(\alpha)}\sum_{i=0}^{n}\int_{t_i}^{t_{i+1}} (t_{n+1}-\tau)^{\alpha-1}q(\tau, T(\tau))d\tau. \tag{3.1}$$

Then using the Lagrange polynomial interpolation, within the interval $[t_{i-1}, t_i]$, function $q(\tau, T(\tau))$ can be approximated as

$$q_i(\tau) = \frac{\tau - t_{i-1}}{t_i - t_{i-1}}q\big(t_i, T(t_i)\big) - \frac{\tau - t_i}{t_i - t_{i-1}}q\big(t_{i-1}, T(t_{i-1})\big)$$

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$$= \frac{q\big(t_i, T(t_i)\big)}{t_i - t_{i-1}}(\tau - t_{i-1}) - \frac{q\big(t_{i-1}, T(t_{i-1})\big)}{t_i - t_{i-1}}(\tau - t_i)$$
$$\approx \frac{q(t_i, T_i)}{t_i - t_{i-1}}(\tau - t_{i-1}) - \frac{q(t_{i-1}, T_{i-1})}{t_i - t_{i-1}}(\tau - t_i).$$

Therefore, bringing the above approximation into the equation (3.1) yields the following equation

$$T(t_{n+1}) = T_0 + \frac{1-\alpha}{N(\alpha)} q\big(t_n, T(t_n)\big) + \frac{\alpha}{N(\alpha)}\frac{1}{\Gamma(\alpha)}\sum_{i=0}^{n}\Big(\frac{q(t_i,T_i)}{h}\int_{t_i}^{t_{i+1}}(t_{n+1}-\tau)^{\alpha-1}(\tau - t_{i-1})d\tau - \frac{q(t_{i-1},T_{i-1})}{h}\int_{t_i}^{t_{i+1}}(t_{n+1}-\tau)^{\alpha-1}(\tau - t_i)d\tau\Big). \tag{3.2}$$

Through the fundamental theorem of integration, we can obtain

$$\int_{t_i}^{t_{i+1}}(t_{n+1}-\tau)^{\alpha-1}(\tau - t_{i-1})d\tau = h^{\alpha+1}\frac{(n+1-i)^{\alpha}(n+2+\alpha-i) - (n-i)^{\alpha}(n+2+2\alpha-i)}{\alpha(\alpha+1)},$$
$$\int_{t_i}^{t_{i+1}}(t_{n+1}-\tau)^{\alpha-1}(\tau - t_i)d\tau = h^{\alpha+1}\frac{(n+1-i)^{\alpha+1} - (n-i)^{\alpha}(n+1+\alpha-i)}{\alpha(\alpha+1)}.$$

For convenient, let

$$X_{\alpha,i,1} = (n+1-i)^{\alpha}(n+2+\alpha-i) - (n-i)^{\alpha}(n+2+2\alpha-i),$$
$$X_{\alpha,i,2} = (n+1-i)^{\alpha+1} - (n-i)^{\alpha}(n+1+\alpha-i).$$

Replacing them to eq. (3.2), we can get the solution in the form as

$$T(t_{n+1}) = T_0 + \frac{1-\alpha}{N(\alpha)} q\big(t_n, T(t_n)\big) + \frac{\alpha}{N(\alpha)}\frac{h^{\alpha}}{\Gamma(\alpha+2)}\sum_{i=0}^{n}\big(q(t_i,T_i)X_{\alpha,i,1} - q(t_{i-1},T_{i-1})X_{\alpha,i,2}\big).$$

The normalization function is taken as follows

$$N(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}.$$

We assign appropriate values to all parameters and take different α values to sketch the solutions of the proposed model (2.2.2). The initial value of the tumor cell group is taken as $T_0 = 0.5$mg/L. The remaining parameter values are taken as λ=1, k=1, c=0.25. We assign the Atangana-Baleanu-Caputo fractional orders of $\alpha = 1$, 0.85, 0.75 and 0.65. And when $\alpha = 1$, it is the ordinary integer order differential equation. The numerical solutions are sketched in Figure 1.

Allee pointed out that swarming is good for population growth and survival, but excessive sparseness and overcrowding can prevent growth and have negative effects on reproduction. As can be seen from Figures 1, the number of cells starts to grow faster and then slowly, eventually reaching a stable level. The solution of the integer-order equation is higher than the solution of the fractional order equation in the later stages, which means that the higher the fractional order, the higher the tumor cell content. In addition, it is found that the solution under the Atangana-Baleanu-Caputo fractional order changes quickly at the beginning and is relatively gently later.

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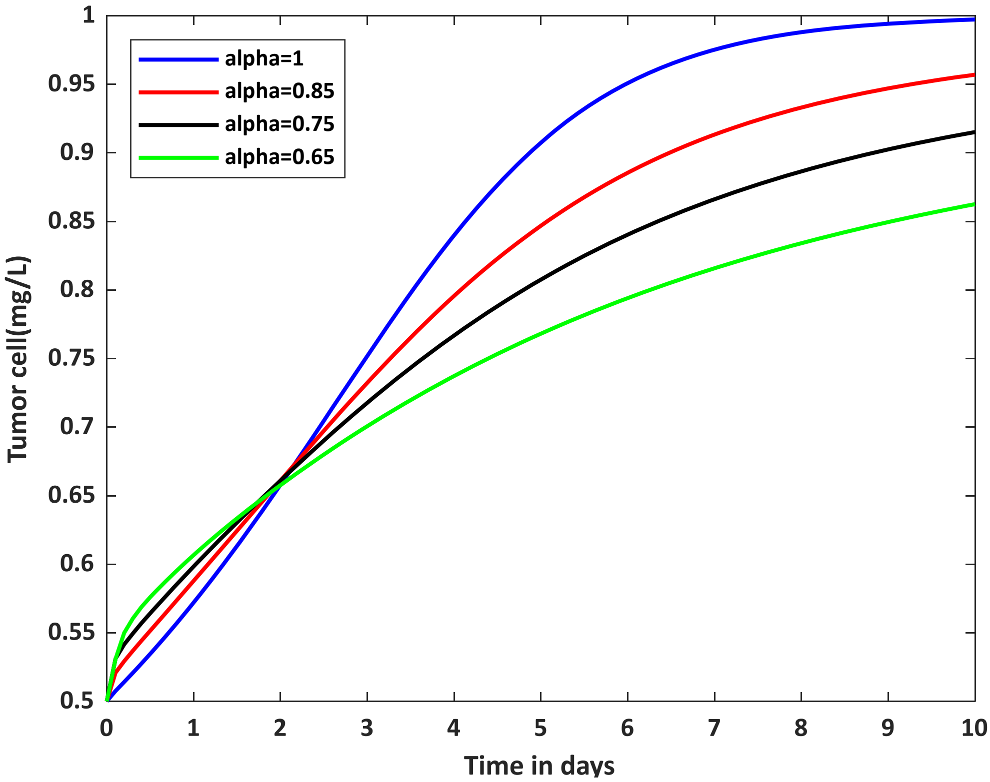

Figure 1. Numerical solutions of the Atangana-Baleanu-Caputo fractional tumor model.

4. CONCLUSION

In this paper, the solution of the tumor growth model with Allee effect with Atanga-Baleanu-Caputo derivative is studied. This derivative has a nonlocal and nonsingular kernel which can effectively describe various processes in the best way. The numerical solution of the model is obtained by the Lagrange polynomial and the fundamental theorem of fractional calculus. And the solutions under different fractional orders are compared intuitively through image. We can find a curve that best fits the actual data by changing the number of fractional orders. This feature is not achievable with integer-order differential equations. This lays the groundwork for exploring the potential impact of the Allee effect in tumor cell populations.

REFERENCES

[1] N. Sene: SIR epidemic model with Mittag-Leffler fractional derivative, Chaos Solitons Fractals, Vol. 137(2020), p.109833.

[2] M. Bachraoui, K. Hattaf, N. Yousfi: Dynamics of a fractional order HBV infection model with capsids and CTL immune response, Commun Math Biol Neurosci, Vol. 2019(2019)No.6, p.1-15.

[3] A. Boukhouima, K. Hattaf, N. Yousfi: In: Mondaini, R. (eds) Trends in Biomathematics: Mathematical Modeling for Health, Harvesting, and Population Dynamics (Springer, Cham, 2019), p. 271-297.

[4] B. Ghanbari, S. Djilali: Mathematical and numerical analysis of a three-species predator-prey model with herd behavior and time fractional-order derivative, Math Methods Appl Sci, Vol. 43 (2020)No.4, p.1736–1752.

[5] R. Ghanbari, D. Kumar: Numerical solution of predator-prey model with Beddington-DeAngelis functional response and fractional derivatives with Mittag-Leffler kernel, Chaos Interdiscip J Nonlinear Sci, Vol. 29(2019)No.6, p.063103.

[6] S. Kumar, R. Kumar, C. Cattani, B. Samet: Chaotic behaviour of fractional predator-prey dynamical system, Chaos Solitons Fractals, Vol. 135(2020)No.17, p.109811.

ISSN: 2472-3703 DOI: 10.6911/WSRJ.202305_9(5).0011

[7] B. Acay, E. Bas, T. Abdeljawad: Fractional economic models based on market equilibrium in the frame of different type kernels, Chaos Solitons Fractals, Vol. 130(2020)No.C, p.109438.

[8] G. Rudolf, F. Mainardi: Fractional Calculus: Integral and Differential Equations of Fractional Order, ArXiv: Math Phys, (2008), p.223-276.

[9] BS.S. Bayin: Definition of the Riesz derivative and its application to space fractional quantum mechanics, J Math Phys, Vol. 57 (2016)No.12, p.123501.

[10] C. Friedrich, H. Schiessel, et al: Constitutive behavior modeling and fractional derivatives, Rheol Ser, Vol.8(1999), p.429-466.

[11] I.M. Sokolov, J. Klafter, et al: Fractional Kinetics, Phys Today, Vol. 55 (2002)No.11, p.48-54.

[12] A. Atangana: New Fractional Derivatives with Nonlocal and Non-Singular Kernel: Theory and Application to Heat Transfer Model, Therm Sci, Vol. 20 (2016).

[13] M. Toufik, A. Atangana: New numerical approximation of fractional derivative with non-local and non-singular kernel: Application to chaotic models, Eur Phys J, Vol. 132(2017)No.444, p.1-16.

[14] M. Al-Refai: On weighted Atangana–Baleanu fractional operators. Adv Differ Equ, Vol. 2020(2020) No.3, p.1-11.

[15] S. Khajanchi, M. Sardar, J. Nieto: Application of Non-singular Kernel in a Tumor Model with Strong Allee Effect, Differ Equ Dyn Syst, (2022).

[16] M. M. Meerschaert, C. Tadjeran: Finite difference approximations for fractional advection-dispersion flow equation, J Comput Appl Math, Vol. 172 (2004)No.1, p.65-77.

[17] D. Kai, et al: A Predictor-Corrector Approach for the Numerical Solution of Fractional Differential Equations, Non Dyn, Vol. 29 (2013), p.3-22.

[18] P. Guo: The Adomian Decomposition Method for a Type of Fractional Differential Equations, J Appl Math Phys, Vol. 7(2019)No.10, p.2459-2466.

[19] F. A. Rihan: On Fractional Order Cancer Model, Adv Calc Var, Vol. 3 (2012)No.2, p.1-6.

[20] L. Bolton, A.H. Cloot, S.W. Schoombie, J.P. Slabbert: A proposed fractional-order Gompertz model and its application to tumour growth data, Math Med Biol, Vol. 32(2015)No.2, p.187–209.

[21] M.A. Dokuyucu, E. Celik, H. Bulut, H.M. Baskonus: Cancer treatment model with the Caputo-Fabrizio fractional derivative, Eur Phys J Plus, Vol. 133(2018)No.92, p.1-6.

[22] C. A. Valentim, N. A. Oliveira, José A. Rabi, Sergio A. David: Can fractional calculus help improve tumor growth models? J Comput Appl Math, Vol. 379(2020), p.1-15.