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A Boubaker Plynomial Collocation Method for Solving Linear Volterra-fredholm Integral Equation in Two-Dimensional Space

Jintong Yang

School of Mathematics and Information, China West Normal University, Nanchong, Sichuan, China

Abstract

This paper presents a collocation method for solving two-dimensional linear volterra fredholm integral equation based on boubaker polynomials. The main characteristic of this method is that it is simple and convenient to calculate, and easy to obtain approximate solutions. The main idea of this method is to simplify the solution of integral equation to the solution of algebraic equation, and discuss the existence, uniqueness and convergence analysis of the solution of this method. Finally, the feasibility and effectiveness of this method were demonstrated through numerical examples.

Keywords

Two-dimensiona linear volterra-fredholm integral equations; Boubaker plynomial; Existence and uniqueness of solutions; Convergence analysis.

1. INTRODUCTION

Integral equation has a long history, and many problems such as mathematical physics can be expressed by integral equation. Due to the different problems, different forms of integral equation will appear. Such as integro differential equations, partial differential integral equation, integral equation and mixed integral equation. In the fields of biology[1] and mechanics[2], integral equation is used to establish the model and solve it. In this paper, the two-dimensional mixed linear volterra-fredholm integral equation is considered:

$$g(s,t)=f(s,t)+\lambda_1\int_a^t k(s,y)g(y,t)dy+\lambda_2\int_a^c q(t,x)g(x,s)dx\,,\quad a\le t\le c \tag{1}$$

where $f(s,t)$, $k(s,y)$, $q(t,x)$ are known kernel functions and λ_1, λ_2 are nonnegative number. $g(s,t)$ is the function to be solved on $\Omega=[a,c]\times[a,c]$.

Due to the development of the times, a single integral equation can not satisfy more complex models. So many scholars turn their attention to the mixed integral equation. The mixed integral equation includes volterra integral and fredholm integral. for a single volterra integral and fredholm integral, the complexity of solving the mixed integral equation increases. How to solve the approximate solution of mixed integral equation is the focus of current research.

The numerical methods commonly used to solve volterra-fredholm integral equation include collocation method[3-11], decomposition method[12-14], approximation method[15-17]and iterative method[18-19]. For the approximation method, its numerical results are quite good, and the error gradually decreases as the number of iterations increases. This method can also be applied to other types of integral equation and integral equation with more complex kernels. The characteristic of the decomposition method is that the obtained solution is in the form of a

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series, which has good convergence and is easy to calculate. Compared with the previous methods, the advantage of the collocation method is that it can quickly, conveniently and effectively find the approximate solution of the integral equation.

Collocation method is a commonly used numerical method, which is flexible and can solve mixed integral equation according to different polynomials. Therefore, it has important practical significance in solving the approximate solution of mixed integral equation.

The organizational structure of this article is as follows: Section 2 introduces the boubaker polynomial. Section 3 introduces the boubaker polynomial collocation method, Section 4 provides an analysis of the existence, uniqueness, and convergence of the solution, and Section 5 provides numerical examples to demonstrate the feasibility and effectiveness of this method. Finally, a brief summary is provided in Section 6.

2. PRELIMINARIES

The standard boubaker polynomial $B_i(x)$, for $i=0,1,\ldots,N$ are defined[20], as

$$B_0(x)=1,$$

$$B_1(x)=x,$$

$$B_2(x)=x^2+2,$$

$$\vdots$$

$$B_N(x)=xB_{N-1}(x)-B_{N-2}(x),\; N\geq 2$$

where N is nonnegative number.

3. DESCRIPTION OF THE METHOD

In this section, the approximate solution of the two-dimensional linear volterra-fredholm integral equation is solved by using the boubaker polynomial collocation method. Expand the unknown function of equation (1) into a boubaker polynomial series with finite terms:

$$g(s,t)\approx g_{nm}(s,t)=\sum_{n=1}^{N+1}\sum_{m=1}^{M+1}d_{nm}B_n(s)B_m(t)=B(s,t)D, \tag{2}$$

where $B(s,t)$ and D are $(N+1)(M+1)\times 1$ matrix, the $B(s,t)$ form as follow

$B(s,t)=[b_{11}(s,t),\ldots,b_{1(M+1)}(s,t),b_{21}(s,t),\ldots,b_{2(M+1)}(s,t),\ldots,b_{(N+1)1}(s,t),\ldots,b_{(N+1)(M+1)}(s,t)]$,

Where $b_{nm}(s,t)=B_n(s)B_m(t)$, $n=1,2,\ldots,N+1$, $m=1,2,\ldots,M+1$.

The D form as follow $D=[d_{11},\ldots,d_{1(M+1)},d_{21},\ldots,d_{2(M+1)},\ldots,d_{(N+1)1},\ldots,d_{(N+1)(M+1)}]^T$,

where d_{nm}, $n=1,2,\ldots,N+1$, $m=1,2,\ldots,M+1$ are the unknown boubaker polynomials coefficients. N and M are nonnegative number.

For the solution of Eq.(1),substitute $g_{nm}(s,t)$ into the Eq.(1) to get

$$g_{nm}(s,t)=f(s,t)+\lambda_1\int_a^t k(s,y)g_{nm}(y,t)dy+\lambda_2\int_a^c q(t,x)g_{nm}(x,s)dx. \tag{3}$$

Convert Eq.(3) into the matrix form

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$$B(s,t)D-\lambda_1\int_a^t k(s,y)B(y,t)Ddy-\lambda_2\int_a^c q(t,x)B(s,x)Ddx=f(s,t), \tag{4}$$

Next, choose nodes as collocation points (s_i,t_j) , $i=1,2,\ldots,N+1$, $j=1,2,\ldots,M+1$. substitute it into Eq.(4) to get

$$B(s_i,t_j)D-\lambda_1\int_a^{t_j} k(s_i,y)B(y,t_j)Ddy-\lambda_2\int_a^c q(t_j,x_i)B(s,x_i)Ddx=f(s_i,t_j) \tag{5}$$

For the above Eq.(5), rewrite it

$(B-K-Q)D=F$

Where

$$B=\begin{bmatrix} B(s_1,t_1), \\ \vdots \\ B(s_1,t_{M+1}), \\ B(s_2,t_1), \\ \vdots \\ B(s_2,t_{M+1}), \\ \vdots \\ B(s_{N+1},t_1), \\ \vdots \\ B(s_{N+1},t_{M+1}) \end{bmatrix}, \qquad F=\begin{bmatrix} f(s_1,t_1), \\ \vdots \\ f(s_1,t_{M+1}), \\ f(s_2,t_1), \\ \vdots \\ f(s_2,t_{M+1}), \\ \vdots \\ f(s_{N+1},t_1), \\ \vdots \\ f(s_{N+1},t_{M+1}) \end{bmatrix},$$

$$K=\begin{bmatrix} \int_a^{t_1} k(s_1,y)B(y,t_1)dy, \\ \vdots \\ \int_a^{t_{M+1}} k(s_1,y)B(y,t_{M+1})dy, \\ \int_a^{t_1} k(s_2,y)B(y,t_1)dy, \\ \vdots \\ \int_a^{t_{M+1}} k(s_2,y)B(y,t_{M+1})dy, \\ \vdots \\ \int_a^{t_1} k(s_{N+1},y)B(y,t_1)dy, \\ \vdots \\ \int_a^{t_{M+1}} k(s_{N+1},y)B(y,t_{M+1})dy \end{bmatrix}, \qquad Q=\begin{bmatrix} \int_a^c q(t_1,x)B(s_1,x)dx, \\ \vdots \\ \int_a^c q(t_{M+1},x)B(s_1,x)dx, \\ \int_a^c q(t_1,x)B(s_2,x)dx, \\ \vdots \\ \int_a^c q(t_{M+1},x)B(s_2,x)dx, \\ \vdots \\ \int_a^c q(t_1,x)B(s_{N+1},x)dx, \\ \vdots \\ \int_a^c q(t_{M+1},x)B(s_{N+1},x)dx \end{bmatrix}.$$

By solving algebraic Eq.(5) we can get the vector D. Approximate the solution of Eq.(1) with substitutiong D in Eq.(2).

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4. UNIQUENESS AND CONVERGENCE ANALYSIS

In this section, we use the following theorem to consider the uniqueness and convergence analysis of the above method:

Theorem1: $g(s,t)$ is the exact solution of eq.(1) of the integral equation, $g_{nm}(s,t)$ is the approximate solution of eq.(1) of the integral equation. There exists a constant α that satisfies the condition of $0<\alpha<1$, therefore, the solution of equation (1) exists and is unique.

Proof: Firstly, use the method of proof to prove the uniqueness of the solution to eq.(1), assuming $g_{nm}(s,t)$ and $\tilde{g}_{nm}(s,t)$ are solutions to eq.(1):

$$\begin{aligned}
\left|g_{nm}(s,t)-\tilde{g}_{nm}(s,t)\right| &\le \left|\lambda_1\int_a^t k(s,y)(g_{nm}(y,t)-\tilde{g}_{nm}(y,t))dy\right|+\left|\lambda_2\int_a^c q(t,x)(g_{nm}(x,s)-\tilde{g}_{nm}(x,s))dx\right| \\
&\le |\lambda_1|\left|\int_a^t k(s,y)(g_{nm}(y,t)-\tilde{g}_{nm}(y,t))dy\right|+|\lambda_2|\left|\int_a^c q(t,x)(g_{nm}(x,s)-\tilde{g}_{nm}(x,s))dx\right| \\
&\le |\lambda_1|\left|\int_a^t |k(s,y)|\left|g_{nm}(y,t)-\tilde{g}_{nm}(y,t)\right|dy\right|+|\lambda_2|\left|\int_a^c |q(t,x)|\left|g_{nm}(x,s)-\tilde{g}_{nm}(x,s)\right|dx\right| \\
&\le |\lambda_1|K\left|\int_a^t \left|g_{nm}(y,t)-\tilde{g}_{nm}(y,t)\right|dy\right|+|\lambda_2|Q\left|\int_a^c \left|g_{nm}(x,s)-\tilde{g}_{nm}(x,s)\right|dy\right| \\
&= \alpha\left|g_{nm}(s,t)-\tilde{g}_{nm}(s,t)\right|
\end{aligned}$$

Where $K=\sup\limits_{(s,t)\in\Omega}\left|\int_a^t k(s,y)dy\right|$, and $Q=\sup\limits_{(s,t)\in\Omega}\left|\int_a^t q(t,x)dx\right|$. So

$$(1-\alpha)\left|g_{nm}(s,t)-\tilde{g}_{nm}(s,t)\right|\le 0$$

Because $\alpha=(|\lambda_1|K+|\lambda_2|Q)(c-a)<1$, and $n,m\to\infty$, $\left|g_{nm}(s,t)-\tilde{g}_{nm}(s,t)\right|=0$, the solution to equation (1) exists and is unique.

Theorem2: $g(s,t)$ is the exact solution of eq.(1) of the integral equation, $g_{nm}(s,t)$ is the approximate solution of eq.(1) of the integral equation. There exists a constant β that satisfies the condition of $0<\beta<1$, so that the solution to eq.(1) is convergent.

Proof: prove the convergence of eq.(1). According to the definition of norm, there are

$$\begin{aligned}
\|g-g_{nm}\| &\le |\lambda_1|\left\|\int_a^t |k(s,y)|\left|g(y,t)-g_{nm}(y,t)\right|dy\right\|_\infty+|\lambda_2|\left\|\int_a^c |q(t,x)|\left|g(x,s)-g_{nm}(x,s)\right|dx\right\|_\infty \\
&\le |\lambda_1|\left\|\int_a^t |k(s,y)|dy\right\|_\infty \|g-g_{nm}\|_\infty+|\lambda_2|\left\|\int_a^c |q(t,x)|dx\right\|_\infty \|g-g_{nm}\|_\infty \\
&= \left(|\lambda_1|\left\|\int_a^t |k(s,y)|dy\right\|_\infty+|\lambda_2|\left\|\int_a^c |q(t,x)|dx\right\|_\infty\right)\|g-g_{nm}\|_\infty \\
&= \beta\|g-g_{nm}\|_\infty
\end{aligned}$$

So

$$(1-\beta)\|g-g_{nm}\|_\infty\le 0$$

When $\beta=|\lambda_1|\left\|\int_a^t |k(s,y)|dy\right\|_\infty+|\lambda_2|\left\|\int_a^c |q(t,x)|dx\right\|_\infty<1$, and $n,m\to\infty$, it can be obtained that

$$\lim_{n,m\to\infty}\|g-g_{nm}\|_\infty=0$$

The convergence proof has been completed.

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5. NUMERICAL EXPERIMENTS

This section gives several examples of two-dimensional linear volterra-fredholm integral equation. To illustrate the feasibility and rationality of this method, we believe that the absolute error between the exact solution and the existing solution is defined as $erroe = |g(s,t) - g_{nm}(s,t)|$, $(s,t) \in \Omega$, where $g_{nm}(s,t)$ and $g(s,t)$ are approximate solution and exact solution respectively.

5.1. Example1

Consider the two-dimensional linear volterra-fredholm integral equation in the form of eq.(1)

$$g(s,t) = f(s,t) + \lambda_1 \int_a^t k(s,y) g(y,t) dy + \lambda_2 \int_a^c q(t,x) g(x,s) dx$$

Where $f(s,t) = t^6 - 5st^5 + \frac{5}{3}t^4 - \frac{5}{2}st^3 + (s + \frac{s(24s^3+12)}{6})t - \frac{s(3s^3+2)}{6}$, $a=0$, $c=1$, $\lambda_1 = 1$, $\lambda_2 = 1$, $k(s,y) = 5s - 2y$, $q(t,x) = -4t + x$, the exact solution is $g(s,t) = st + t^4$.

Table 1. Results for example1 with n=m=3,4,5

(s,t)	exact solution	n=m=3 approximate solution	n=m=4 approximate solution	n=m=5 approximate solution
(0,0)	0	0	0	-3.039790641423679e-12
(0.1,0.1)	0.0101	2.199800000000023e-02	1.010000000000021e-02	1.009999999696148e-02
(0.2,0.2)	0.0416	5.155200000000013e-02	4.159999999999983e-02	4.159999999697339e-02
(0.3,0.3)	0.0981	1.006620000000000e-01	9.810000000000024e-02	9.809999999701999e-02
(0.4,0.4)	0.1856	1.813279999999997e-01	1.855999999999999e-01	1.855999999971414e-01
(0.5,0.5)	0.3125	3.055499999999998e-01	0.3.125	3.124999999973938e-01
(0.6,0.6)	0.4896	4.853280000000000e-01	4.896000000000000e-01	4.895999999978499e-01
(0.7,0.7)	0.7301	7.326620000000003e-01	7.301000000000001e-01	7.300999999985969e-01
(0.8,0.8)	1.0496	1.059552000000000e+00	1.0496	1.049599999999740e+00
(0.9,0.9)	1.4661	1.477998000000000e+00	1.466100000000000e+00	1.466100000001400e+00

Table 2. Error for example1 with n=m=3,4,5

(s,t)	n=m=3	n=m=4	n=m=5
(0,0)	0	0	3.039790641423679e-12
(0.1,0.1)	1.189800000000023e-02	2.116362640691705e-16	3.038517354392312e-12
(0.2,0.2)	9.952000000000120e-03	1.734723475976807e-16	3.026613681900159e-12
(0.3,0.3)	2.561999999999939e-03	2.081668171172169e-16	2.980046764911037e-12
(0.4,0.4)	4.272000000000331e-03	1.665334536937735e-16	2.858657754956084e-12
(0.5,0.5)	6.950000000000234e-03	0	2.606193039156324e-12
(0.6,0.6)	4.272000000000165e-03	1.110223024625157e-16	2.150279954094003e-12
(0.7,0.7)	2.562000000000064e-03	1.110223024625157e-16	1.403321903126198e-12
(0.8,0.8)	9.952000000000183e-03	0	2.600142323672117e-13
(0.9,0.9)	1.189799999999974e-02	2.220446049250313e-16	1.399769189447397e-12

5.2. Example2

Solving two-dimensional linear volterra-fredholm integral equation in the form of eq.(1)

$$g(s,t) = f(s,t) + \lambda_1 \int_a^t k(s,y) g(y,t) dy + \lambda_2 \int_a^c q(t,x) g(x,s) dx ,$$

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Where $f(s,t)=\frac{s^4}{4}+\frac{st^7}{3}+st^4$, $k(s,y)=sy$, $q(t,x)=x^2$, $a=0$, $c=1$, $\lambda_1=\lambda_2=-1$,the exact solution is $g(s,t)=st^4$.

Table 3. Results for example2 with n=m=3,4,5

(s,t)	exact solution	n=m=3 approximate solution	n=m=4	n=m=5
(0.1,0.1)	0	2.550452000000000e-01	1.000000000001000e-05	9.999997957699236e-06
(0.2,0.2)	0.00032	4.848256000000000e-01	3.199999999999870e-04	3.199999959153654e-04
(0.3,0.3)	0.0024	6.884784000000002e-01	2.430000000000043e-03	2.429999993873166e-03
(0.4,0.4)	0.0102	8.699408000000001e-01	1.024000000000003e-02	1.023999999183078e-02
(0.5,0.5)	0.0313	1.037950000000000e+00	3.125000000000000e-02	3.124999998978850e-02
(0.6,0.6)	0.0778	1.206043200000000e+00	7.776000000000005e-02	7.775999998774630e-02
(0.7,0.7)	0.1681	1.392557600000000e+00	1.680699999999999e-01	1.680699999857038e-01
(0.8,0.8)	0.3277	1.620630400000000e+00	3.276800000000002e-01	3.276799999836617e-01
(0.9,0.9)	0.5905	1.918198800000000e+00	5.904900000000002e-01	5.904899999816196e-01
(1,1)	1	2.318000000000000e+00	1.000000000000000e+00	9.999999999795770e-01

Table 4. Error for example2 with n=m=3,4,5

(s,t)	n=m=3	n=m=4	n=m=5
(0.1,0.1)	2.550352000000000e-01	9.998376909389761e-18	2.042300766572297e-12
(0.2,0.2)	4.845056000000000e-01	1.311884628707460e-17	4.084634648744701e-12
(0.3,0.3)	6.860484000000001e-01	4.163336342344337e-17	6.126835150332965e-12
(0.4,0.4)	8.597008000000000e-01	2.428612866367530e-17	8.169218773668163e-12
(0.5,0.5)	1.006700000000000e+00	0	1.021149831359480e-11
(0.6,0.6)	1.128283200000000e+00	0	1.225375356739278e-11
(0.7,0.7)	1.224487600000000e+00	1.387778780781446e-16	1.429625862137129e-11
(0.8,0.8)	1.292950400000000e+00	1.110223024625157e-16	1.633837509729119e-11
(0.9,0.9)	1.327708800000000e+00	0	1.838063035108917e-11
(1,1)	1.318000000000000e+00	0	2.042299662718960e-11

6. CONCLUDING

In this paper, we propose a boubaker polynomial collocation method for solving two-dimensional linear mixed integral equation. The feasibility and effectiveness of this method were verified through numerical examples. In the analysis of the example results in Section 5, it was found that when the exact solution appears in the form of a polynomial, the approximate solution obtained using the method proposed in this paper performs better. And when the values of n, m are large, the approximate solution obtained using this method is better than the approximate solution obtained by taking smaller values of n, m. However, if the obtained approximate solution is close enough to the exact solution, even if the value of n, m increases, its effect will not change significantly, and sometimes it may even weaken. However, by comparing the errors, suitable values of n, m can be found in order to obtain the best approximate solution.

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